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Linear And Nonlinear Analytical Modeling of Laminated Composite Beams In Three Points Bending

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Abstract: *The large current development of aerospace and automotive technologies is based on the use of composite materials which provide significant weight savings compared to their mechanical characteristics. Correct dimensioning of composite structures requires a thorough knowledge of their behavior in small as in large deflection.This work aims to simulate linear and nonlinear behavior of laminates composites under three-point bending test. The used modelization is based on first-order shear deformation theory (FSDT), classical plate theory (CPT) and Von-Karman's equations for large deflection. A differential equation of Riccati, describing the variation of the deflection depending on the load, was obtained. Hence, the results deduced show a good correlation with experimental curves.*

Keywords: *linear behavior ; nonlinear behavior ; laminate; Graphite epoxy; laminated beam; Three-point bending; Geometric nonlinearity ; failure ; macroscopic curve.*

I. Introduction

Today, the Composites materials are used practically in all industries and still undergoing strong expansion rate. They are the source of large challenges in various high technology achievements. But the continuation of the development of their use in structures requires to establish the necessary tools for modeling their mechanical behavior and their dimensioning at failure. These models are validated with the results obtained by experimental testing. The biaxial testing is the ideal test to validate the macroscopic behavior of composite structures. However, this type of test is not often used, it is difficult to achieve and very expensive. Thus, an alternative to biaxial testing is given by the bending tests, which help to follow the damage progression of specimen to the final failure for some configurations. Many authors such as Gustavo [1], Xiwen [2] and Moreno [3] were interested in this test. In our case, we use the experimental results developed by Echaabi [4] with observed various behaviors by varying the distance between the supports and the geometrical dimensions of the specimens (Fig 3).

The nonlinearity depends on the thickness ratio *L/h*, the orthotropy, boundary conditions and the number of laminate layers. A small ratio *L/h* leads to a linear behavior and the specimens with a large ratio present nonlinear responses. Our study is limited only to the thickness ratio *l/h* (Fig 3) [7].

Irhirane [5, 26] used two formulations an analytical method with transverse shearing and finite element method. The first-order shear deformation theory (FSDT), which takes into account the transverse shearing strain and the correction of coefficients, remains the best approach to characterize and simulate analytically the macroscopic curves of failure and the sequences of failure for test specimens A and D (Table.1 and Fig 4). On the other hand, the use of the finite element method significantly improves the results on the level of the breaking loads and the flexural stiffness [5]. Indeed, the results of Irhirane work permit to obtain a good correlation with experimental curves only in the linear behavior. A nonlinear behavior observed in other specimens, has not been studied and still difficult to resolve. In this respect, our work is focused on modeling the behavior of specimens with a nonlinear response.

In the literature, there are many theories for modeling the nonlinear behavior of laminated composite. These theories can be divided into three main categories [8]: The Equivalent Single Layer (ESL), Layer-Wise (LW) [9] and Zig-Zag. [10] Three other theories are deduced from ESL: the classical plate theory (CPT), the first-order shear deformation theory which takes into account the transverse shearing strain (FSDT) and higher-order shear deformation theories (HSDTs). Other simplified theories have been recently developed with less unknowns [11, 12]. Previous theories are usually used to analyze the elastic behavior of industrial structures. In our case, we used the classical plate theory, which has the advantage of using less unknowns and which describes with good accuracy the fields of stresses and strains in the laminated composite beams with a large thickness ratio *L/h* in three points bending [11].

The Von-Karman's large deflection theory leads to a good modeling of the nonlinear behavior of the laminated composite. This theory is used by many authors such as Padmanav Dash and B. N. Singh [7] JN

Reddy [15] Y.X. Zhang and K. S. Kim [6] and H. Nguyen-Van [14]. The introduction of this theory has allowed us to develop a differential equation of Riccati. The results are given in detail in this paper and also compared with experimental curves.

II. Basic Formulations

A typical laminated composite graphite/epoxy beam with n layers is shown in Fig. 1**.**

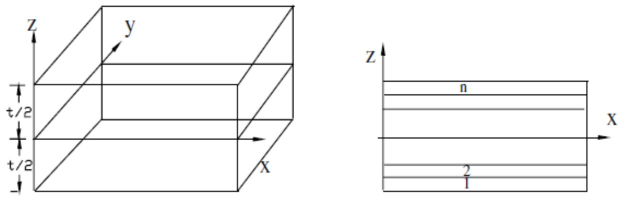

Fig 1: graphite epoxy beam with N layers.

The elastic mechanical behavior of a structure constituted of composite materials is generally modeled by the first-order shear deformation theory. The FSDT proposed by Reissner and Mindlin [8] accounts for shear deformation effects by the way of linear variation of in-plane displacements through the thickness. Since the FSDT violates the equilibrium conditions on the top and bottom surfaces of the plate, a shear correction factor is required to compensate for the difference between actual stress state and assumed constant stress state [8]:

$$\begin{cases} u(x,y,z) = u^0(x,y) + z\,\theta_x(x,y) \\ v(x,y,z) = v^0(x,y) + z\,\theta_y(x,y) \\ w(x,y,z) = w^0(x,y) \end{cases} \quad (1)$$

Where *u, v* and *w* are displacements of the specimen in directions *x, y* and *z* respectively.
u^0, *v0 and w0* are displacements in plane of membrane and θ_x and θ_y rotations around axes *x* and *y*.
The strain can be written as:

$$\varepsilon = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \quad (2)$$

Substituting Eq.(1) into (2). The strain field is given by the following relationship:

$$\varepsilon = \varepsilon_m + z\varepsilon_b \quad (3)$$

with:
The membrane strain field can be rewritten as:

$$\varepsilon_m = \begin{Bmatrix} \dfrac{\partial u^0}{\partial x} \\ \dfrac{\partial v^0}{\partial y} \\ \dfrac{\partial u^0}{\partial y} + \dfrac{\partial v^0}{\partial x} \end{Bmatrix} \quad (4)$$

The bending strain field is:

$$\varepsilon_b = \begin{Bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x} \end{Bmatrix} \quad (5)$$

The transverse shear strain is given as:

$$\gamma = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \theta_x + \dfrac{\partial w}{\partial x} \\ \theta_y + \dfrac{\partial w}{\partial y} \end{Bmatrix} \tag{6}$$

The CPT model is based on the Kirchhoff–Love hypothesis that the straight lines remain straight and perpendicular to the midplane after deformation. These assumptions imply the vanishing of the shear and normal strains, and consequently, neglecting the shear and normal deformation effects [8]. This theory gives good result with fewer unknowns for a large thickness ratio *l/h* [11] see Fig N°2.

$$\begin{cases} u(x,y,z) = u^0(x,y) - z\dfrac{\partial w^0}{\partial x} \\ v(x,y,z) = v^0(x,y) - z\dfrac{\partial w^0}{\partial y} \\ w(x,y,z) = w^0(x,y) \end{cases} \tag{7}$$

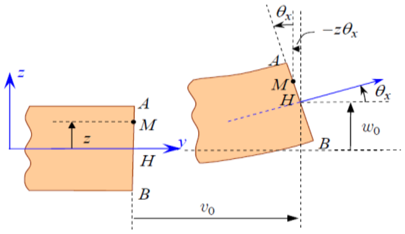

Fig 2: The classical plate theory (CPT) [18].

For large deformation analysis, the in-plane vector of Green strain at any point in a beam element is:

$$\varepsilon^* = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial v}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial x}\right)^2\right] \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}\left[\left(\dfrac{\partial u}{\partial y}\right)^2 + \left(\dfrac{\partial v}{\partial y}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2\right] \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix} \tag{8}$$

Substituting Eq. (7) into (8) and considering the Von Karman large deflection assumption, the strain can be written as:

$$\varepsilon^* = \varepsilon_m^* - zk \tag{9}$$

The membrane strain field ε_m^* can be rewritten as:

$$\varepsilon_m^* = \varepsilon_m^{*L} + \varepsilon_m^{*NL} \tag{10}$$

Where ε_m^{*L} is the linear part and ε_m^{*NL} is nonlinear part of the membrane strain field:

$$\varepsilon_m^{*L} = \begin{Bmatrix} \dfrac{\partial u^0}{\partial x} \\ \dfrac{\partial v^0}{\partial y} \\ \dfrac{\partial u^0}{\partial y} + \dfrac{\partial v^0}{\partial x} \end{Bmatrix} \qquad \varepsilon_m^{*NL} = \begin{Bmatrix} \dfrac{1}{2}\left[\left(\dfrac{\partial w^0}{\partial x}\right)^2\right] \\ \dfrac{1}{2}\left[\left(\dfrac{\partial w^0}{\partial y}\right)^2\right] \\ \dfrac{\partial w^0}{\partial x}\dfrac{\partial w^0}{\partial y} \end{Bmatrix} \tag{11}$$

The bending strain field is:

$$k = \begin{Bmatrix} \dfrac{\partial^2 w^0}{\partial x^2} \\ \dfrac{\partial^2 w^0}{\partial y^2} \\ 2\dfrac{\partial^2 w^0}{\partial x \partial y} \end{Bmatrix} \tag{12}$$

For laminated composite beams, the constitutive relationship can be expressed as [5] and [18-20]:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & F_{44} & F_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & F_{45} & F_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{13}$$

Aij (*i, j = 1, 2, 6*): The in plane stiffness matrix.
Dij (*i, j = 1, 2, 6*): The bending stiffness matrix.
Bij (*i, j = 1, 2, 6*): The bending–extensional coupling stiffness.
Fij (*i, j = 4,5*): The transverse shear stiffness.
Mx , *My* et *Mxy*: The resulting bending moment.
Ny , *Nx* et *Nxy*: The resulting membrane force.
Q_x et Q_y: The resulting transverse shear force.

III. Application And Analysis

In this study, the material used is an epoxy graphite laminate with a layer sequence of $[[+45/-45/90/0]_3]_s$. Dimensions of the test specimen are given in Table 1. Its mechanical characteristics are:
E11=116* GPa, *E22= E33= 6.9* GPa, *G12=5.6* GPa, *G13=3.4* GPa, *G23=2.5* GPa, *ν12= ν13= ν23= 0.3,

Table 1: The Geometrical characteristics of the specimen considered in mm.

Specimen	Length L	Distance *l* between supports	width b	Thickness h	Thickness ratio *l/h*
A	75	57.5	25	3.6	16
B	150	115.0	25	3.6	32
C	150	136.5	25	3.6	38
D	75	57.5	10	3.6	16
E	150	115.0	10	3.6	32
F	150	136.5	10	3.6	38

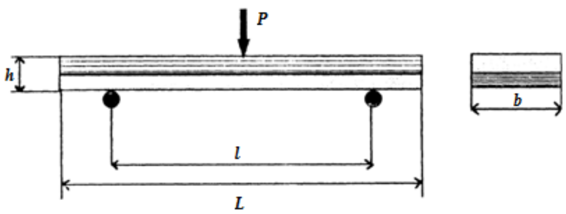

Fig 3: Experimental setup [4].

The small thickness ratio *l/h*=16 for specimens A and D allows us to predict a linear behavior of the variation of the center deflection w_c depending on the load *P* [7] (Fig N°4). The Modeling is based on a first degree scheme which takes into account the transverse shearing strain and the correction of coefficients. The center deflection is given by the following formulation [5, 26].

$$w_c = \frac{Pl^3}{48b} D_{11}^* \left[1 + 12 \frac{F_{55}^*}{D_{11}^*} \frac{1}{l^2} \right] \tag{14}$$

w_c : The center deflection.
P : The applied load.
b : The width of the specimen.
D_{11}^* : Element in reverse bending stiffness matrix.
L : The length of the specimen.
F_{55}^* : Element in reverse transverse shear stiffness matrix.

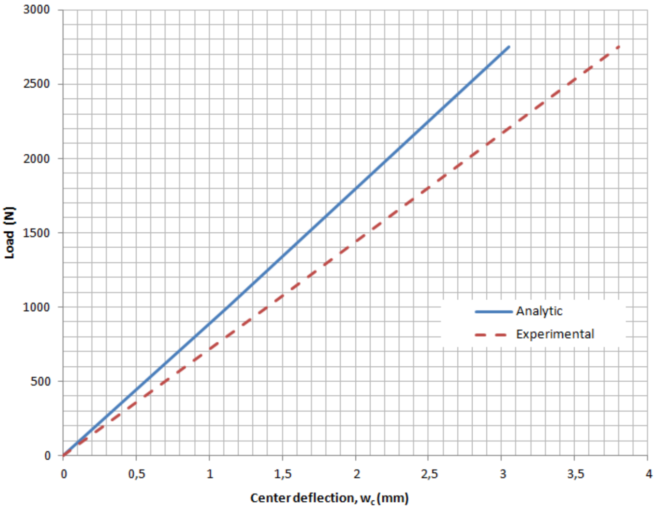

Fig 4: variation of the center deflection w_c according to the load P of the test specimen A

Otherwise, the specimens E and F (with a relatively large ratio l/h=32 and l/h=38) allows us to predict a nonlinear behavior of the variation of the center deflection w_c depending on the load P [7]. So, we start with the classical plate theory (CPT).

$$u(x,y,z) = u^0(x,y) - z\frac{\partial w^0}{\partial x} \tag{15}$$

The strain field is given by the following equation:

$$\varepsilon_x^* = \frac{\partial u^0}{\partial x}(x,y) - z\frac{\partial^2 w^0}{\partial x^2}(x,y) \tag{16}$$

For modeling of the nonlinear behavior, we introduce the Von Karman theory. The equation (16) becomes:

$$\varepsilon_x^* = \frac{\partial u^0}{\partial x}(x,y) - z\frac{\partial^2 w^0}{\partial x^2}(x,y) + \frac{1}{2}\left(\frac{\partial w^0}{\partial x}\right)^2 \tag{17}$$

On the other hand, the constitutive relationship of a symmetrical laminated composite in bending can be expressed as:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x^0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} \tag{18}$$

Where the bending strain field are defined as:

$$\kappa^0 = \begin{Bmatrix} \kappa_x^0 \\ \kappa_y^0 \\ \kappa_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial^2 w^0}{\partial x^2} \\ \dfrac{\partial^2 w^0}{\partial y^2} \\ 2\dfrac{\partial^2 w^0}{\partial x \partial y} \end{Bmatrix} \tag{19}$$

In three points bending test, the strain field is defined as:

$$\varepsilon_x^* = -zD_{11}^* M_x \tag{20}$$

Where D_{11}^* is an element in reverse bending stiffness matrix.

By (17) and (20) we deduce the variation of the deflection according to moment:

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{2z}\left(\frac{\partial w}{\partial x}\right)^2 + D_{11}^* M_x \tag{21}$$

Introducing the bending moment : $M = bM_x$, knowing that : $M = \frac{Px}{2},\ 0 \le x \le \frac{l}{2}$. Therefore equation (21) becomes:

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{2z}\left(\frac{\partial w}{\partial x}\right)^2 + D_{11}^* \frac{Px}{2b} \tag{22}$$

With the following boundary conditions: $w(0) = 0$, $\frac{\partial w}{\partial x}\left(\frac{l}{2}\right) = 0$

After, we introduce a change variable:

$$y = \frac{\partial w}{\partial x} \tag{23}$$

Therefore the equation (22) becomes:

$$y' = cy^2 + dx \tag{24}$$

This is a differential equation of Riccati which has a solution according to the Cauchy-Lipschitz if the coefficients $c = \frac{1}{2z}$ *et* $d = D_{11}^* \frac{P}{2b}$ are continuous. This equation models the nonlinear behavior of laminated beams with a relatively large thickness ratio *l/h* in three points bending. The exact solution is given by the following formulation [16, 17]:

$$y = -\frac{1}{c}\frac{F'(x)}{F(x)},\ \text{with}\, F(x) = \sqrt{x}\left[C_1 J_{1/3}\left(\frac{2}{3}\sqrt{cd}\,x^{3/2}\right) + C_2 Y_{1/3}\left(\frac{2}{3}\sqrt{cd}\,x^{3/2}\right)\right] \tag{25}$$

Where $J_m(z)$ and $Y_m(z)$ are the Bessel functions, C_1 and C_2 are constants.
Therefore, the variation of the deflection w_c according to the load P is given by the following formulation:

$$w = -\frac{1}{c}\ln\left[\frac{2}{3}\sqrt{cx}\sqrt{d}\left(C_1 J_{1/3} x^{3/2} + C_2 Y_{1/3} x^{3/2}\right)\right] + C_3 \tag{26}$$

Where C_3 is a constant.
The analytical results obtained from equation (26) are given in Fig.5 and Fig.6. A good correlation was observed between the experimental and analytical results.

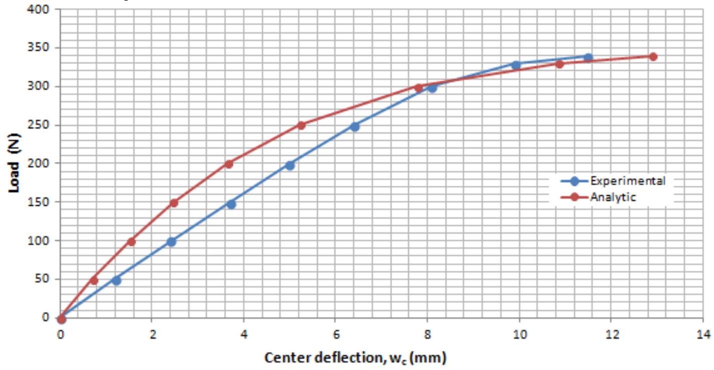

Fig.5: variation of the center deflection w_c according to the load P of the test specimen E.

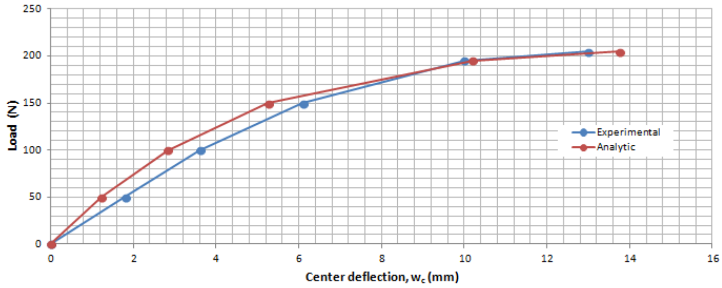

Fig.6 variation of the center deflection w_c according to the load P of the test specimen F.

IV. Results And Discussion

Many authors such as Y.X. Zhang and Y.K. Cheung (2003) [21] V. Loc Tran & al (2015) [22] and H. Nguyen-Van (2014), [14] have developed analytical and numerical approaches to link the state of the nonlinear behavior bending with the main characteristics of the specimens. These methods have complexity in application with a large number of unknowns to be determined. Therefore, we propose an analytical modeling of three points bending of symmetrical laminated composites beams with less unknowns. The experimental macroscopic curves show a linear behavior that has been modeled by Irhirane work. However, the nonlinear behavior observed for some values of the thickness ratio *L/h* was not modeled untill today. In this work, we used the classical plate theory (CPT) and Von Karman theory to study the nonlinear behavior of the beams and we deduced that this nonlinearity can be modeled by a Riccati equation. The results predict the experimental nonlinear curves with excellent accuracy. Indeed, the Riccati equations were used to model many physical phenomena. For example, in classical mechanics [23], in population dynamics [24] and a large variety application in automatic [25]. The results of our analytical modelization in the case of nonlinear behavior improve significantly the previously obtained ones which are based on a numerical resolution and only model the linear behavior. However, the material undergoes a successive damage before the first macroscopic failure. Thus, the microscopic failures may be observed in the matrix but not in the fiber. The following of this work is to introduce the failure criteria to predict the failure mode and stress associated with the first macroscopic failure thus to study the impact of material damage to the beam behavior before the first failure.

V. Conclusion

All the results obtained with all the proposed approximations, prove the difficulty of modeling the behavior of laminated beams in bending. Whereas, the Riccati equation deduced from the classical plate theory (CPT) and Von Karman theory used in this article improves the results compared to those in the literature.
Furthermore, the thickness ratio *L/h* is a primary factor in geometrically nonlinear bending. In fact, the laminated composite beams with a small ratio exhibit a linear behavior. Accordingly, the modeling is recommended according to the first-order shear deformation theory which takes into account the transverse shearing strain and the correction of coefficients. However, the laminated composite beams with a large thickness ratio *L/h* show a nonlinear behavior, the modelization according to the classical plate theory (CPT), is strongly recommended because of its simple in application (3 unknowns) and acceptable results.

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