Research Inventy: International Journal of Engineering And Science
Vol.6, Issue 4 (April 2016), PP -102-105
Issn (e): 2278-4721, Issn (p):2319-6483, www.researchinventy.com

On Convergence of Jungck Type Iteration for Certain Contractive Conditions

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***Abstract*:** In this article we prove the strong convergence result for a pair of nonself mappings using Jungck S- iterative scheme in Convex metric spaces satisfying certain contractive condition. The results are the generalization of some existing results in the literature.
Keywords: *Iterative schemes, contractive condition, Convex metric spaces.*

I. INTRODUCTION AND PRELIMINARIES

In 1970, Takahashi [16] introduced the notion of convex metric space and studied the fixed point theorems for nonexpansive mappings. He defined that a map $W: X^2 \times [0,1] \to X$ is a convex structure on X if

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y)$$

for all $x, y, u \in X$ and $\lambda \in [0,1]$. A metric space (*X*, *d*) together with a convex structure *W* is known as convex metric space and is denoted by (X, d, W). A nonempty subset *C* of a convex metric space is convex if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$.

Remark 1.1 Every normed space is a convex metric space, where a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for all $x, y, z \in X$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$. In fact,

$$\begin{aligned} d(u, W(x, y, z; \alpha, \beta, \gamma)) &= \left\| u - (\alpha x + \beta y + \gamma z) \right\| \\ &\leq \alpha \left\| u - x \right\| + \beta \left\| u - y \right\| + \gamma \left\| u - z \right\| \\ &= \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z), \end{aligned}$$

for all $u \in X$. But there exists some convex metric spaces which cannot be embedded into normed spaces.

Let X be a Banach space, Y an arbitrary set, and $S, T: Y \to X$ such that $T(\mathrm{Y}) \subseteq \mathrm{S}(\mathrm{Y})$. For $x_0 \in Y$, consider the following iterative scheme:

$$Sx_{n+1} = Tx_n, \; n = 0,1,2 \ldots. \tag{1.1}$$

is called Jungck iterative scheme and was essentially introduced by Jungck [1] in 1976 and it becomes the Picard iterative scheme when $S = I_d$ (identity mapping) and *Y* = *X*.

For $\alpha_n \in [0,1]$, Singh et al. [2] defined the Jungck-Mann iterative scheme as

$$Sx_{n+1} = (1-\alpha_n) \mathrm{S}\, x_n + \alpha_n T x_n, \; n = 0,1,2 \ldots. \tag{1.2}$$

For $\alpha_n, \beta_n, \gamma_n \in [0,1]$, Olatinwo defined the Jungck Ishikawa [3] (see also [4, 5]) and Jungck-Noor [6] iterative schemes as

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n) \mathrm{S}\, x_n + \alpha_n T y_n, \\ Sy_n &= (1-\beta_n) \mathrm{S}\, x_n + \beta_n T x_n, \; n = 0,1,2 \ldots. \end{aligned} \tag{1.3}$$

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n) \mathrm{S}\, x_n + \alpha_n T y_n, \\ Sy_n &= (1-\beta_n) \mathrm{S}\, x_n + \beta_n T z_n, \\ Sz_n &= (1-\gamma_n) \mathrm{S}\, x_n + \gamma_n T x_n, \; n = 0,1,2 \ldots. \end{aligned} \tag{1.4}$$

respectively.

Jungck Agarwal et al. [18] iteration is given as:

$$Sx_{n+1} = (1-\alpha_n)\,\mathrm{T}\,\mathrm{x}_n + \alpha_n T y_n, \qquad (1.5)$$
$$Sy_n = (1-\beta_n)\,\mathrm{S}\,x_n + \beta_n T x_n$$

And Agarwal et al. [12] iterative scheme is given as:

$$x_{n+1} = (1-\alpha_n)\,\mathrm{T}\,\mathrm{x}_n + \alpha_n T y_n,$$
$$y_n = (1-\beta_n) x_n + \beta_n T x_n \qquad (1.6)$$

Remark 1.2 If $X = Y$ *and* $S = I_d$ (identity mapping), then the Jungck-Noor (1.4), Jungck-Ishikawa (1.3), Jungck-Mann (1.2) and Jungck Agarwal et al.(1.5) iterative schemes become the Noor [9], Ishikawa [10] ,Mann [11] and the Agarwal et al.iterative [12] schemes respectively.

Jungck [1] used the iterative scheme (1.1) to approximate the common fixed points of the mappings S and T satisfying the following Jungck contraction:

$d\,(Tx, Ty) \leq \mathrm{d}(Sx, Sy)$, $0\leq\alpha<1$. (1.7)

Olatinwo [3] used the following more general contractive definition than (1.7) to prove the stability and strong convergence results for the Jungck-Ishikawa iteration process:

(a) There exists a real number $a \in [0,1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ such that $\phi(0) = 0$ and for all $x, y \in Y$, we have

$$d\,(Tx, Ty) \leq \varphi\, d\,(Sx, Tx) + a\, d\,(Sx, Sy). \qquad (1.8)$$

(b) There exists a real number $M \geq 0$, $a \in [0,1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ such that $\phi(0) = 0$ and for all $x, y \in Y$, we have

$$d\,(Tx, Ty) \leq \frac{\varphi\, d\,(Sx, Tx) + a\, d\,(Sx, Sy)}{1 + M\, d\,(\mathrm{S}\,\mathrm{x}, \mathrm{T}\,\mathrm{x})} \qquad (1.9)$$

Now we give the above iterative schemes in the setting of convex metric spaces:

Let $(\mathrm{X}, \mathrm{d}, \mathrm{W})$ be a convex metric spaces. For $x_0 \in X$, we have

(1.1.1) Jungck Picard iterative scheme:

$$Sx_{n+1} = Tx_n,\ n = 0,1,2\ldots.$$

(1.1.2) Jungck Mann iterative scheme:

$$Sx_{n+1} = W\,(\mathrm{S}\,x_n, Tx_n, \alpha_n),\ n = 0,1,2\ldots.$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in [0,1].

(1.1.3) Jungck Ishikawa iterative scheme:

$$Sx_{n+1} = W\,(\mathrm{S}\,x_n, Ty_n, \alpha_n)$$
$$Sy_n = W\,(\mathrm{S}\,x_n, Tx_n, \beta_n),\ n = 0,1,2\ldots.$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences in [0,1].

(1.1.4) Jungck Noor iterative scheme:

$$Sx_{n+1} = W\,(\mathrm{S}\,x_n, Ty_n, \alpha_n)$$
$$Sy_n = W\,(\mathrm{S}\,x_n, Tz_n, \beta_n)$$
$$Sz_n = W\,(\mathrm{S}\,x_n, Tx_n, \gamma_n),\ n = 0,1,2\ldots.$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are real sequences in [0,1].

(1.1.5) Jungck Agarwal iterative scheme:

$$Sx_{n+1} = W\,(Tx_n, Ty_n, \alpha_n)$$
$$Sy_n = W\,(\mathrm{S}\,x_n, Tx_n, \beta_n),\ n = 0,1,2\ldots.$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in [0,1].

(1.1.6) Jungck Agarwal iterative scheme:

$$x_{n+1} = W(Tx_n, Ty_n, \alpha_n)$$
$$y_n = W(x_n, Tx_n, \beta_n),\ n = 0,1,2\ldots.$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive numbers in [0,1].

Definition 1.4 (see [14, 15]). Let f and g be two self-maps on X. A point x in X is called (1) a fixed point of f if $(x) = x$; (2) coincidence point of a pair (f, g) if $fx = gx$; (3) common fixed point of a pair (f, g) if $x = fx = gx$. If $w = fx = gx$ for some x in X, then w is called a point of coincidence of f and g. A pair (f, g) is said to be weakly compatible if f and g commute at their coincidence points.

Now we will give our main results:

II. CONVERGENCE RESULTS

Theorem 2.1. Let (X, d, W) be an arbitrary Convex metric space and let $S, T : Y \to X$ be nonself – operators on an arbitrary set Y satisfying contractive condition (1.8) , (1.9). Assume that $T(Y) \subseteq S(Y)$, $S(Y)$ is a complete subspace of X and $Sz = Tz = p$ (say). Let $\varphi : R^+ \to R^+$ be monotone increasing function such that $\varphi(0) = 0$. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck –Agarwal et. al iteration process defined by (1.1.5), where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers in $[0,1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, the Jungck –Agarwal et. al iterative process $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. Also, p will be the unique common fixed point of S,T provided that Y=X, and S and T are weakly compatible.

Proof. First, we prove that z is the unique coincidence point of $S\ and\ T$
by using condition (1.8). Let $C(S,T)$ be the set of the coincidence points of $S\ and\ T$. Suppose that there exists $z_1, z_2 \in C(S,T)$ such that $Sz_1 = Tz_1 = p_1$ and $Sz_2 = Tz_2 = p_2$. If $p_1 = p_2$, then $Sz_1 = Sz_2$ and since S is injective, it follows that $z_1 = z_2$. If $p_1 \neq p_2$, then from condition (1.8), for mappings S and T, we have

$$\begin{aligned} 0 \le d(p_1, p_2) &= d(Tz_1, Tz_2) \\ &\le \varphi d(Sz_1, Tz_2) + a d(Sz_1, Sz_2) \\ &= a d(p_1, p_2) \end{aligned}$$

which implies that $(1-a)d(p_1, p_2) \le 0$. So we have $(1-a) > 0$,

Since $a \in [0,1)$, but $d(p_1, p_2) \le 0$, which is as contradiction since norm is negative. So we have $d(p_1, p_2) = 0$, that is $p_1 = p_2 = p$. Since $p_1 = p_2$, then we have that $p_1 = Sz_1 = Tz_1 = Sz_2 = Tz_2 = p_2$, leading to $Sz_1 = Sz_2 \Rightarrow z_1 = z_2 = z$.

Hence z is unique coincidence point of S and T.

Now we prove that iterative process $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

Using condition (1.8) and (1.9), we have

$$\begin{aligned} d(Sx_{n+1}, p) &= d(W(Tx_n, Ty_n, \alpha_n), p) \\ &\le (1-\alpha_n) d(Tx_n, p) + \alpha_n d(Ty_n, p) \\ &= (1-\alpha_n) d(Tz, Tx_n) + \alpha_n d(Tz, Ty_n) \\ &\le (1-\alpha_n)\left[\frac{\varphi d(Sz, Tz) + a d(Sz, Sx_n)}{1 + M d(Sz, Tz)}\right] + \alpha_n \left[\frac{\varphi d(Sz, Tz) + a d(Sz, Sy_n)}{1 + M d(Sz, Tz)}\right] \\ &\le (1-\alpha_n) d(Sx_n, p) + \alpha_n a d(Sy_n, p) \qquad (2.1.1) \end{aligned}$$

For $d(Sy_n, p)$, we have

$$
\begin{aligned}
d(Sy_n, p) &= d(W(Sx_n, Tx_n, \beta_n), p) \\
&\leq (1-\beta_n)\, d(Sx_n, p) + \beta_n d(Tx_n, p) \\
&\leq (1-\beta_n)\, d(Sx_n, p) + \beta_n \left[\frac{\varphi d(Sz, Tz) + a d(Sz, Sx_n)}{1 + M d(Sz, Tz)} \right] \\
&\leq (1-\beta_n + a\beta_n)\, d(Sx_n, p) \qquad (2.1.2)
\end{aligned}
$$

From (2.1.1) and (2.1.2), we get

$$
\begin{aligned}
d(Sx_{n+1}, p) &\leq (1-\alpha_n)\, d(Sx_n, p) + \alpha_n a(1-\beta_n + a\beta_n)\, d(Sx_n, p) \\
&= a[1 - \alpha_n \beta_n (1-a)]\, d(Sx_n, p) \\
&\leq [1 - \alpha_n (1-a)]\, d(Sx_n, p) \\
&\leq \prod_{k=0}^{n} [1 - (1-a)\alpha_k]\, d(Sx_0, p) \\
&\leq e^{-(1-a)\sum_{k=0}^{\infty} \alpha_k} d(Sx_0, p). \qquad (2.1.3)
\end{aligned}
$$

Since $\alpha_k \in [0,1]$, $0 \leq a < 1$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, so

$\leq e^{-(1-a)\sum_{k=0}^{n} \alpha_k} d(Sx_0, p) \to 0$ as $n \to \infty$.

Hence from equation (2.1.3) we get, $d(Sx_{n+1}, p) \to 0$ as $n \to \infty$, that is $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

Corollary 2.2. If we take S=I (Identity mapping) then the iterative scheme (1.1.6) becomes Agarwal et. al iteration as defined by (1.1.7). Convergence of Agarwal et. al iterative scheme can be proved on the same lines as in Theorem 2.1.

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