Research Inventy: International Journal of Engineering And Science
Vol.4, Issue 8 (August 2014), PP 49-55
Issn (e): 2278-4721, Issn (p):2319-6483, www.researchinventy.com

Convergence Theorems for Implicit Iteration Scheme With Errors For A Finite Family Of Generalized Asymptotically Quasi-Nonexpansive Mappings In Convex Metric Spaces

Renu Chugh[1] and Preety[2] and Sanjay Kumar [3]
[1,2,3]Department of Mathematics, Maharshi Dayanand University, Rohtak-124001(INDIA)

ABSTRACT: *In this paper, we prove the strong convergence of an implicit iterative scheme with errors to a common fixed point for a finite family of generalized asymptotically quasi-nonexpansive mappings in convex metric spaces. Our results refine and generalize several recent and comparable results in uniformly convex Banach spaces. With the help of an example we compare implicit iteration used in our result and some other implicit iteration.*

KEYWORDS: *Implicit iteration process with errors, Generalized asymptotically quasi-nonexpansive mappings, Common fixed point, Strong convergence, Convex metric spaces.*

I. INTRODUCTION AND PRELIMINARIES

In 1970, Takahashi [1] introduced the notion of convexity in metric space and studied some fixed point theorems for nonexpansive mappings in such spaces.

Definition 1.1 [1] A map $W : X^2 \times [0,1] \to X$ is a convex structure in X if

$$d(u,W(x,y,\lambda)) \le \lambda d(u,x) + (1-\lambda)d(u,y)$$

for all $x, y, u \in X$ and $\lambda \in [0,1]$. A metric space (X,d) together with a convex structure W is known as convex metric space and is denoted by (X,d,W). A nonempty subset C of a convex metric space is convex if $W(x,y,\lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0,1]$. All normed spaces and their subsets are the examples of convex metric spaces.

Remark 1.2 Every normed space is a convex metric space, where a convex structure $W(x,y,z;\alpha,\beta,\gamma) = \alpha x + \beta y + \gamma z,$ for all $x, y, z \in X$ and $\alpha,\beta,\gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1.$ In fact,

$$\begin{aligned} d(u,W(x,y,z;\alpha,\beta,\gamma)) &= \|u-(\alpha x+\beta y+\gamma z)\| \\ &\le \alpha\|u-x\| + \beta\|u-y\| + \gamma\|u-z\| \\ &= \alpha d(u,x) + \beta d(u,y) + \gamma d(u,z), \end{aligned}$$

for all $u \in X.$ But there exists some convex metric spaces which cannot be embedded into normed spaces.

Throughout this paper, we assume that X is a metric space and let $F(T_i)\ (i \in N)$ be the set of all fixed points of mappings T_i respectively, that is, $F(T_i) = \{x \in X : T_i x = x\}.$ The set of common fixed points of $T_i\ (1,2,......N)$ denoted by $\mathrm{F} = \bigcap_{i=1}^{N} F(T_i).$

Definition 1.3 [2, 3] Let $T : X \to X$ be a mapping and F(T) denotes the fixed point of T. Then the mapping T is said to be

(1) nonexpansive if

$$d(Tx,Ty) \le d(x,y),\ \ \forall x, y \in X.$$

(2) quasi-nonexpansive if $F(T) \ne \phi$ and

$$d(Tx,p) \le d(x,p),\ \ \forall x \in X,\ \ \forall p \in F(T).$$

(3) asymptotically nonexpansive [4] if there exists a sequence $\{u_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} u_n = 0$ such that

$$d(T^n x, T^n y) \le (1+u_n) d(x,y),\ \forall x, y \in X \text{ and } n \ge 1.$$

(4) asymptotically quasi-nonexpansive if $F(T) \ne \phi$ and there exists a sequence $\{u_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} u_n = 0$ such that

$$d(T^n x, p) \le (1+u_n) d(x,p),\ \forall x \in X,\ p \in F(T) \text{ and } n \ge 1.$$

(5) generalized asymptotically quasi-nonexpansive [5] if $F(T) \ne \phi$ and there exist two sequences of real numbers $\{u_n\}$ and $\{c_n\}$ with $\lim_{n\to\infty} u_n = 0 = \lim_{n\to\infty} c_n$ such that

$$d(T^n x, p) \le (1+u_n) d(x,p) + c_n,\ \forall x \in X,\ p \in F(T) \text{ and } n \ge 1.$$

(6) Uniformly L-Lipschitzian if there exists a positive constant L such that

$$d(T^n x, T^n y) \le L d(x,y),\ \forall x, y \in X \text{ and } n \ge 1.$$

Remark 1.4 From definition 1.2, it follows that if F(T) is nonempty, then a nonexpansive mapping is quasi-nonexpansive and an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. But the converse does not hold. From (5), if $c_n = 0$ for all $n \ge 1$, then T becomes asymptotically quasi-nonexpansive. Hence the class of generalized asymptotically quasi-nonexpansive maps includes the class of asymptotically quasi-nonexpansive maps.

The Mann and Ishikawa iteration processes have been used by a number of authors to approximate the fixed point of nonexpansive, asymptotically nonexpansive mappings and quasi-nonexpansive mappings on Banach spaces (see e.g., [9-16].

In 1998, Xu [9] gave the following definitions: For a nonempty subset C of a normed space E and $T: C \to C$, the Ishikawa iteration process with errors is the iterative sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \qquad (1.1) \\ y_n &= \alpha_n' x_n + \beta_n' T^n x_n + \gamma_n' v_n, \quad n \ge 1, \end{aligned}$$

where $\{u_n\}, \{v_n\}$ are bounded sequence in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n'\}, \{\beta_n'\}, \{\gamma_n'\}$ are sequences in [0,1] such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha_n' + \beta_n' + \gamma_n'$ for all $n \ge 1$.

If $\beta_n' = 0 = \gamma_n'$ for all $n \ge 1$, then (1.3) reduces to Mann iteration process with errors. The normal Ishikawa and Mann iteration processes are special cases of the Ishikawa iteration process with errors.

In 2001, Xu and Ori [6] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space H. Let C be a nonempty subset of H. Let $T_1, T_2, \ldots\ldots, T_N$ be selfmappings of C and suppose that $\mathrm{F} = \bigcap_{i=1}^{N} F(T_i) \ne \phi$, the set of common fixed points of $T_i, i = 1, 2, \ldots.. N$. An implicit iteration process for a finite family of nonexpansive mappings is defined as follows: Let $\{t_n\}$ a real sequence in (0, 1), $x_0 \in C$:

$$\begin{aligned} x_1 &= t_1 x_0 + (1-t_1) T_1 x_1, \\ x_2 &= t_2 x_1 + (1-t_2) T_2 x_2, \\ \ldots &= \ldots., \\ x_N &= t_N x_{n-1} + (1-t_N) T_N x_N, \\ x_{N+1} &= t_{N+1} x_N + (1-t_{N+1}) T_1 x_{N+1}, \\ \ldots. &= \ldots. \end{aligned}$$

which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1-t_n)T_n x_n, \qquad n \ge 1, \qquad (1.2)$$

where $T_k = T_{k(\bmod N)}$. (Here the mod N function takes values in the set {1,2,….N}). They proved the weak convergence of the iterative scheme (1.2).

In 2003, Sun [7] extend the process (1.2) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with $\{\alpha_n\}$ a real sequence in (0,1) and an initial point $x_0 \in C,$ which is defined as follows:

$$\begin{aligned}
&x_1 = \alpha_1 x_0 + (1-\alpha_1)T_1 x_1,\\
&x_2 = \alpha_2 x_1 + (1-\alpha_2)T_2 x_2,\\
&\ldots = \ldots.\\
&x_N = \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N,\\
&x_{N+1} = \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1^2 x_{N+1},\\
&\ldots. = \ldots.\\
&x_{2N} = \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})T_N^2 x_{2N},\\
&x_{2N+1} = \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})T_1^3 x_{2N+1},\\
&\ldots. = \ldots.
\end{aligned}$$

which can be written in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^k x_n, \qquad n \ge 1, \qquad (1.3)$$

Where $n = (k-1)N + i, \quad i \in \{1,2,\ldots.N\}.$

Sun [7] proved the strong convergence of the process (1.3) to a common fixed point in real uniformly convex Banach spaces, requiring only one member T in the family $\{T_i, i = 1,2,\ldots..N\}$ to be semi compact.

Definition 1.5 Let C be a convex subset of a convex metric space (X,d,W) with a convex structure W and let I is the indexing set i.e. $I = \{1,2,\ldots.N\}.$ Let $\{T_i : i \in I\}$ be N generalized asymptotically quasi-nonexpansive mappings on C. For any given $x_0 \in C,$ the iteration process $\{x_n\}$ defined by

$$\begin{aligned}
&x_1 = W(x_0, T_1 x_1, u_1; \alpha_1, \beta_1, \gamma_1),\\
&x_2 = W(x_1, T_2 x_2, u_2; \alpha_2, \beta_2, \gamma_2),\\
&\ldots = \ldots\\
&x_N = W(x_{N-1}, T_N x_N, u_N; \alpha_N, \beta_N, \gamma_N),\\
&x_{N+1} = W(x_N, T_1^2 x_{N+1}, u_{N+1}; \alpha_{N+1}, \beta_{N+1}, \gamma_{N+1}),\\
&\ldots = \ldots.\\
&x_{2N} = W(x_{2N-1}, T_N^2 x_{2N}, u_{2N}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}),\\
&x_{2N+1} = W(x_{2N}, T_1^3 x_{2N+1}, u_{2N+1}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}),\\
&\ldots = \ldots
\end{aligned}$$

where $\{u_n\}$ is a bounded sequence in C and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are sequences in [0,1] such that $\alpha_n + \beta_n + \gamma_n = 1.$

The above sequence can be written in compact form as

$$x_n = W(x_{n-1}, T_i^k x_n, u_n; \alpha_n, \beta_n, \gamma_n), \ \ n \ge 1, \qquad (1.4)$$

with $n = (k-1)N + i, i \in I \ and \ T_n = T_{i(\bmod N)} = T_i$.

Let $\{T_i : i \in I\}$ be N uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive selfmappings of C. Then clearly (1.5) exists.

If $u_n = 0$ in 1.5 then,

$$x_n = W(x_{n-1}, T_i^k x_n; \alpha_n, \beta_n), \ \ n \geq 1, \tag{1.5}$$

with $n = (k-1)N + i, i \in I \textit{ and } T_n = T_{i(\mathrm{mod}\, N)} = T_i$ and $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0,1] such that $\alpha_n + \beta_n = 1.$

The purpose of this paper is to prove the strong convergence of implicit iteration with errors defined by (1.5) for generalized asymptotically quasi-nonexpansive in the setting of convex metric spaces.
The following Lemma will be used to prove the main results:

Lemma 1.6 [8] Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq a_n + b_n, \ \ n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < \infty,$ then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0.$

Now we prove our main results:

2. Main Results

Theorem 2.1 Let C be a nonempty closed convex subset of a complete Convex metric space X. Let $T_i : C \to C \ \ (i \in I = \{1, 2.....N\})$ be N uniformly L-Lipschitzian generalized quasi-nonexpansive mappings with $\{u_{in}\}, \{c_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty.$ Suppose that $\mathrm{F} = \bigcap_{i=1}^{N} F(T_i) \neq \phi.$ Let $\{x_n\}$ be the implicit iteration process with errors defined by (1.4) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2}).$ Then $\lim_{n\to\infty} d(x_n, \mathrm{F})$ exists.

Proof: Let $p \in F = \bigcap_{i=1}^{N} F(T_i),$ using (1.5) and (5), we have

$$\begin{aligned}
d(x_n, p) &= d(W(x_{n-1}, T_i^k x_n, u_n; \alpha_n, \beta_n, \gamma_n), p) \\
&\leq \alpha_n d(x_{n-1}, p) + \beta_n d(T_i^k x_n, p) + \gamma_n d(u_n, p) \\
&\leq \alpha_n d(x_{n-1}, p) + \beta_n [(1 + u_{ik}) d(x_n, p) + c_{ik}] + \gamma_n d(u_n, p) \\
&= \alpha_n d(x_{n-1}, p) + (1 - \alpha_n - \gamma_n)[(1 + u_{ik}) d(x_n, p) + c_{ik}] + \gamma_n d(u_n, p) \\
&\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n)[(1 + u_{ik}) d(x_n, p) + c_{ik}] + \gamma_n d(u_n, p) \\
&\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n)(1 + u_{ik}) d(x_n, p) + (1 - \alpha_n) c_{ik} + \gamma_n d(u_n, p) \\
&\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n + u_{ik}) d(x_n, p) + (1 - \alpha_n) c_{ik} + \gamma_n d(u_n, p)
\end{aligned} \tag{2.1.1}$$

Since $\lim_{n\to\infty} \gamma_n = 0,$ there exists a natural number n_1 such that for $n > n_1, \ \gamma_n < \frac{s}{2}.$

Hence,

$$\alpha_n = 1 - \beta_n - \gamma_n \geq 1 - (1 - s) - \frac{s}{2} = \frac{s}{2} \tag{2.1.2}$$

for $n > n_1.$ Thus, We have from (2.1.1) that

$$\alpha_n d(x_n, p) \leq \alpha_n d(x_{n-1}, p) + u_{ik} d(x_n, p) + (1 - \alpha_n) c_{ik} + \gamma_n d(u_n, p) \tag{2.1.3}$$

and

$$\begin{aligned}
d(x_n, p) &\leq d(x_{n-1}, p) + \frac{u_{ik}}{\alpha_n} d(x_n, p) + \left(\frac{1}{\alpha_n} - 1\right) c_{ik} + \frac{\gamma_n}{\alpha_n} d(u_n, p) \\
&\leq d(x_{n-1}, p) + \frac{2u_{ik}}{s} d(x_n, p) + \left(\frac{2}{s} - 1\right) c_{ik} + \frac{2\gamma_n}{s} d(u_n, p)
\end{aligned} \tag{2.1.4}$$

$$d(x_n,p)\le\left(\frac{s}{s-2\mu_{ik}}\right)d(x_{n-1},p)+\left(\frac{2}{s}-1\right)\left(\frac{s}{s-2\mu_{ik}}\right)c_{ik}+\left(\frac{s}{s-2\mu_{ik}}\right)\frac{2M}{s}\gamma_n$$
$$=\left(1+\frac{2\mu_{ik}}{s-2\mu_{ik}}\right)d(x_{n-1},p)+\left(\frac{2}{s}-1\right)\left(1+\frac{2\mu_{ik}}{s-2\mu_{ik}}\right)c_{ik}+\left(1+\frac{2\mu_{ik}}{s-2\mu_{ik}}\right)\frac{2M}{s}\gamma_n \quad (2.1.5)$$

where $M=\sup_{n\ge1}\{d(u_n,p)\}$, since $\{u_n\}$ is a bounded sequence in C. Since $\sum_{n=1}^{\infty}u_{ik}<\infty$ for all $i\in I$. This gives that there exists a natural number $n_0\ (as\ k>(n_0/N)+1)$ such that $s-2\mu_{ik}>0$ and $\mu_{ik}<s/4$ for all $n>n_0$. Let $v_{ik}=\frac{2\mu_{ik}}{s-2\mu_{ik}}$ and $t_{ik}=\left(\frac{2}{s}-1\right)\left(1+\frac{2\mu_{ik}}{s-2\mu_{ik}}\right)c_{ik}$. Since $\sum_{k=1}^{\infty}u_{ik}<\infty$ and $\sum_{k=1}^{\infty}c_{ik}<\infty$ for all $i\in I$, it follows that $\sum_{k=1}^{\infty}v_{ik}<\infty$ and $\sum_{k=1}^{\infty}t_{ik}<\infty$. Therefore from (2.1.5) we get,

$$d(x_n,p)\le(1+v_{ik})d(x_{n-1},p)+t_{ik}+\frac{2M}{s}(1+v_{ik})\gamma_n \quad (2.1.6)$$

This further implies that

$$d(x_n,F)\le(1+v_{ik})d(x_{n-1},F)+t_{ik}+\frac{2M}{s}(1+v_{ik})\gamma_n \quad (2.1.7)$$

Since by assumptions, $\sum_{k=1}^{\infty}v_{ik}<\infty$, $\sum_{k=1}^{\infty}t_{ik}<\infty$ and $\sum_{k=1}^{\infty}\gamma_{ik}<\infty$. Therefore, applying Lemma (1.6) to the inequalities (2.1.6) and (2.1.7), we conclude that both $\lim_{n\to\infty}d(x_n,p)$ and $\lim_{n\to\infty}d(x_n,\mathrm{F})$ exists.

Theorem 2.2 Let C be a nonempty closed convex subset of a complete Convex metric space X. Let $T_i:C\to C\ \ (i\in I=\{1,2.....N\})$ be N uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with $\{u_{in}\},\{c_{in}\}\subset[0,\infty)$ such that $\sum_{n=1}^{\infty}u_{in}<\infty$ and $\sum_{n=1}^{\infty}c_{in}<\infty$. Suppose that $\mathrm{F}=\bigcap_{i=1}^{N}F(T_i)\ne\phi$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (1.4) with the restrictions $\sum_{n=1}^{\infty}\gamma_n<\infty$ and $\{\beta_n\}\subset(s,1-s)$ for some $s\in(0,\frac{1}{2})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mapping $\{T_i:i\in I\}$ if and only if

$$\liminf_{n\to\infty}d(x_n,\mathrm{F})=0.$$

Proof: From Theorem 2.1 $\lim_{n\to\infty}d(x_n,\mathrm{F})$ exists. The necessity is obvious. Now we only prove the sufficiency. Since by hypothesis $\liminf_{n\to\infty}d(x_n,\mathrm{F})=0$, so by Lemma 1.6, we have

$$\lim_{n\to\infty}d(x_n,\mathrm{F})=0. \quad (2.2.1)$$

Next we prove that $\{x_n\}$ is a Cauchy sequence in C. Note that when $x>0, 1+x\le e^x$. It follows from (2.1.6) that for any $m\ge1$, for all $n\ge n_0$ and for any $p\in\mathrm{F}$, we have

$$d(x_{n+m},p)\le\exp\left[\sum_{i=1}^{N}\sum_{k=1}^{\infty}v_{ik}\right]d(x_n,p)+\sum_{i=1}^{N}\sum_{k=1}^{\infty}t_{ik}$$
$$+\frac{2M}{s}\exp\left[\sum_{i=1}^{N}\sum_{k=1}^{\infty}v_{ik}\right]\sum_{n=1}^{\infty}\gamma_n$$

$$< Qd(x_n, p) + \sum_{i=1}^{N}\sum_{k=1}^{\infty} t_{ik} + \frac{2QM}{s}\sum_{n=1}^{\infty}\gamma_n, \qquad (2.2.2)$$

where, $Q = \exp\left[\sum_{i=1}^{N}\sum_{k=1}^{\infty} v_{ik}\right] + 1 < \infty.$

Let $\varepsilon > 0.$ Also $\lim_{n\to\infty} d(x_n, p)$ exists, therefore for $\varepsilon > 0,$ there exists a natural number n_1 such that $d(x_n, p) < \varepsilon / 6(1+Q),$ $\sum_{i=1}^{N}\sum_{k=1}^{\infty} t_{ik} < \varepsilon / 3$ and $\sum_{n=1}^{\infty}\gamma_n < s\varepsilon / 6QM$ for all $n \ge n_1.$ So we can find $p^* \in \mathrm{F}$ such that $d(x_{n1}, p^*) < \varepsilon / 3(1+Q).$ Hence, for all $n \ge n_1$ and $m \ge 1,$ we have that

$$\begin{aligned} d(x_{n+m}, x_n) &\le d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\le Qd(x_{n1}, p^*) + \sum_{i=1}^{N}\sum_{k=n1}^{\infty} t_{ik} \\ &\quad + \frac{2QM}{s}\sum_{n=n1}^{\infty}\gamma_n + d(x_{n1}, p^*) \\ &= (1+Q)d(x_{n1}, p^*) + \sum_{i=1}^{N}\sum_{k=n1}^{\infty} t_{ik} + \frac{2QM}{s}\sum_{n=n1}^{\infty}\gamma_n \\ &< (1+Q).\frac{\varepsilon}{3(1+Q)} + \frac{\varepsilon}{3} + \frac{2QM}{s}.\frac{s\varepsilon}{6QM} = \varepsilon. \qquad (2.2.3) \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in C. And, the completeness of X implies that $\{x_n\}$ must be convergent. Let us assume that $\lim_{n\to\infty} x_n = p.$ Now, we show that p is common fixed point of the mappings the mappings $\{T_i : i \in I\}.$ And, we know that $\mathrm{F} = \bigcap_{i=1}^{N} F(T_i)$ is closed. From the continuity of $d(x, \mathrm{F}) = 0$ with $\lim_{n\to\infty} d(x_n, \mathrm{F}) = 0$ and $\lim_{n\to\infty} x_n = p,$ we get

$$d(p, \mathrm{F}) = 0, \qquad (2.2.4)$$

and so $p \in F,$ that is p is a common fixed point of the mappings $\{T_i\}_{i=1}^{N}.$

Hence the proof.

If $u_n = 0$, in above theorem, we obtain the following result:

Theorem 2.3 Let C be a nonempty closed convex subset of a complete Convex metric space X. Let $T_i : C \to C \ \ (i \in I = \{1, 2.....N\})$ be N uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with $\{c_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} c_{in} < \infty.$ Suppose that $\mathrm{F} = \bigcap_{i=1}^{N} F(T_i) \ne \phi.$ Let $\{x_n\}$ be the implicit iteration process with errors defined by (1.5) with $\{\alpha_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2}).$ Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mapping $\{T_i : i \in I\}$ if and only if

$$\liminf_{n\to\infty} d(x_n, \mathrm{F}) = 0.$$

From Lemma 1.6 and Theorem 2.2, we can easily obtain the result.

Corollary 2.4 Let C be a nonempty closed convex subset of a complete Convex metric space X. Let $T_i : C \to C \ \ (i \in I = \{1, 2.....N\})$ be N uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with $\{u_{in}\}, \{c_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty.$ Suppose that

$\mathrm{F}=\bigcap_{i=1}^{N} F(T_i) \neq \phi.$ Let $\{x_n\}$ be the implicit iteration process with errors defined by (1.4) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2}).$ Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mapping $\{T_i : i \in I\}$ if and only if there exists a subsequence $\{x_{nj}\}$ of $\{x_n\}$ which converges to p.

Numerical Example: Let $\{T_i\}_{i=1}^{n}$ be the family of generalized asymptotically quasi- non expansive mappings defined by, $T_n x = \dfrac{x}{n+2}, \quad n \geq 1$, we know that for $n \geq 1$, given family is non expansive and since zero is the only common fixed point of given family, so it is quasi-nonexpansive and so generalized asymptotically quasi-nonexpansive. Now the initial values used in C++ program for formation of our result are x_0=.5 and $\alpha_n \to 0$, we have following observations about the common fixed point of family of generalized asymptotically quasi non expansive mappings and with help of these observations we prove that the implicit iteration (1.2) and (1.3) has same rate of convergence and converges fast as $\alpha_n \to 0$ .From the table given below we can prove the novelty of our result.

Tabular observations of Implicit Iteration (1.2) and (1.3), where initial approximation is x_0==0.5.

N	0	1	2	3	---	---	27	28	29	30
x_n for implicit iteration (1.2)	1e-011	1.5e-022	2e-033	2.5e-044	---	---	1.45e-307	1.5e-318	0	0
x_n for implicit k-step iteration (1.3)	5.02488e-012	5.04988e-023	5.075e-034	5.10025e-045	---	---	5.74538e-309	5.77415e-320	0	0

REFERENCES

[1] W. Takahashi: A convexity in metric space and nonexpansive mappings I, Kodia Math. Sem. Rep., vol. 22, (1970), 142-149.

[2] J. K. Kim, K. H. Kim and K. S. Kim: Convergence theorems of modified three-step iterative sequences with mixed errors for asymptotically quasi-nonexpansive mappings in Banach spaces, Panamer. Math. J., vol. 14, (2004) 1, 45-54.

[3] J. K. Kim, K. H. Kim and K. S. Kim: Three-step iterative sequences with errors for asymptotically quasi-nonexpansive mappings in convex metric spaces, Nonlinear Anal. Convex Anal. RIMS, vol. 1365, (2004), 156-165.

[4] K. Goebel and W. A. Kirk: A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc., vol. 35, (1972), 171-174.

[5] S. Imnang and S. Suantai: Common fixed points of multi-step Noor iterations with errors for a finite family of generalized asymptotically quasi-nonexpansive mappings, Abstr. Appl. Anal., vol. 2009.

[6] H. K. Xu and R. G. Ori: An implicit iteration process for nonexpansive mappings, Numer. Funct. Anal. Optim., vol. 22, (2001), 767-773.

[7] Z. H. Sun: Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings, J. Math. Anal. Appl., vol. 286, (2003), 351-358.

[8] K. K. Tan and H. K. Xu: Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl., vol. 178, (1993), 301-308.

[9] Y. Xu: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive operator equations, J. Math. Anal. Appl., vol. 224, (1998), 91-101.

[10] M. K. Ghosh and L. Debnath: Convergence of Ishikawa iterates of quasi-nonexpansive mappings, J. Math. Anal. Appl., vol. 207, (1997), 96-103.

[11] S. H. Khan and W. Takahashi: Approximating common fixed points of two asymptotically nonexpansive mappings, Sci. Math. Jpn., vol. 53, (2001), 143-148.

[12] L. S. Liu: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, J. Math. Anal. Appl., vol. 194, (1995), 114-125.

[13] Q. H. Liu: Iterative sequence for asymptotically quasi-nonexpansive mappings, J. Math. Anal. Appl., vol. 259, (2001), 1-7.

[14] Q. H. Liu: Iterative sequence for asymptotically quasi-nonexpansive mappings with error member, J. Math. Anal. Appl., vol. 259, (2001), 18-24.

[15] W. V. Petryshyn and T. E. Williamson: Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings, J. Math. Anal. Appl., vol. 43, (1973), 459-497.

[16] T. Shimizu and W. Takahashi, Strong convergence theorem for asymptotically nonexpansive mappings, Nonlinear Anal., vol. 26, (1996), 265-272.