Research Inventy: International Journal Of Engineering And Science
Vol.4, Issue 3(March 2014), PP 01-06
Issn (e): 2278-4721, Issn (p):2319-6483, www.researchinventy.com

The Methods of Solution for Constrained Nonlinear Programming

[1]Chiranjib Mukherjee and [2]Dr.Gyan Mukherjee
[1]*Research Scholar in Computer Science, Magadh University(INDIA)*
[2]*Department of Mathematics, S.P.S. College, B.R.A.B.University (INDIA)*

Abstract: We shall be concerned with the problem of determining a solution of constrained nonlinear programming problems. The approach here is to replace a constrained problem with one that is unconstrained. The reduced problem is then solved using an iterative technique – “Barrier Function Method” and “Quadratic Penalty Function Method”.

Key words: Barrier function Method, Quadratic Penalty Function Method”, Interior Point, Equality Constraints, Objective function, Feasible Solutions, Lagrangian-penalty function method.

I. INTRODUCTION: BARRIER FUNCTION METHOD AND QUADRATIC PENALTY FUNCTION METHOD

We shall consider only such problems which have the form

$$\text{Minimize } f(x) \text{ subject to } g_i(x) \leq 0 \text{ for } i = 1, 2, \ldots m. \tag{1}$$

Note that problem (1) does not contain any equality constraints. We shall assume that function f is continuous over the set F = {x : each $g_i(x) \leq 0$}, and that g_i, g_{i+1}, …… g_m are continuous over $R_{n\times 1}$. Moreover, we shall assume that F has a nonempty interior and that each boundary point F is an accumulation point of the interior of F. This means that each boundary point of F can be approached via the interior point of F.
Some common barrier functions for problem (1) are

$$b(x) = - \sum_{i=1}^{m} 1 / g_i(x) \tag{2}$$

and

$$b(x) = \sum_{i=1}^{m} In \mid g_i(x) \mid \tag{3}$$

Note that b(x) is, in either case, continuous throughout the interior of F. Moreover, $b(x) \to \infty$ as x approaches the boundary of F via the interior of F. Rather solve (1), we intend to solve the following problem:

$$\text{Minimize } f(x) + 1 / \beta\, b(x) \text{ subject to each } g_i(x) < 0 \tag{4}$$

where $\beta > 0$.

As an example:

Minimize x subject to $x \geq 5$.

Solution: A barrier function of the type (3) will be used to simplify computations. In particular, let $\beta > 0$ and solve the problem:

$$\text{Minimize } x - 1 / \beta \text{ In } |5 - x| \text{ subject to } x > 5. \tag{5}$$

Problem (5) can be solved in any standard fashion. The minimum value of the objective function occurs when $x = 5 + 1 / \beta$ and is equal to

$$5 + 1/\beta - 1/\beta \text{ In } 1/\beta \qquad (6)$$

Note that for each $\beta > 0$, x is larger than 5 and approaches 5 as $\beta \to \infty$. Since $\lim_{\beta \to \infty} -(1/\beta) \text{ In } (1/\beta) = 0$, it also follows that (6) approaches the maximum value of 5 for f as $\beta \to \infty$.

Since most practical problems have bounded variables, we shall again assume that the set of feasible solutions, F, of problem (1) is bounded. It should be noted that the bounds may or may not be explicitly displayed in the constraints $g_i(x) = 0$. Nevertheless, with this assumption F is both closed and bounded.

In Quadratic Penalty Function Method, we shall again discuss the solution of the problem.

$$\text{Minimize } f(x) \text{ subject to } h_i(x) = 0, \text{ for } i = 1, 2, \ldots, m \qquad (7)$$

where f, h_1, h_2, ….,h_m are now continuously differentiable. We shall again assume that the set F, of feasible solutions of (7) is nonempty. The continuity of the h_i ensures that F is closed. Since most practical problems have bounds imposed on their variables, we shall assume that F is also bounded. As before, the Weierstras's theorem guarantees the existence of a solution, x*, of problem (7).

Compute vectors x* and λ* that satisfy.

$$0 = \frac{\partial L}{\partial x}(x^*, \lambda^*) = \nabla f(x^*)^T + \lambda^{*T} \frac{\partial L}{\partial x}(x^*)$$

and

$$0 = \frac{\partial L}{\partial x}(x^*, \lambda^*) = h(x^*) \qquad (8)$$

Where $L(x, \lambda) = \lambda^T h(x)$ and $h(x) = [h_1 \ldots\ldots h_m]^T$. But the system of equations (8) is difficult to solve. To overcome this difficulty a method which combines the penalty function method and Langrange's method was devised for solving (8). This new method could be called Lagrangian-penalty function method. However, it is often referred to as multiplier methods. We restrict our analysis to a particular method called the quadratic penalty function method. Our approach is some intuitive.

Consider a solution x* of (7). Let λ* be the corresponding vector of Lagrange multipliers for which equations (8) hold. Note that whenever $x \in F$, then

$$L(x^*, \lambda^*) = f(x^*) \le h(x) = f(x) + \lambda^{*T} h(x^*) = L(x^*, \lambda^*)$$

Thus, $\min\{ L(x^*, \lambda^*) : x \in F\} = L(x^*, \lambda^*)$ and

$$\min \{f(x): x \in F\} = \min \{ L(x^*, \lambda^*) : x \in F\} \qquad (9)$$

This suggests that rather than solve (7) we could solve the problem on the right side of (9), possibly using a penalty function method.

i.e.

$$\text{Minimize } f(x) + \lambda^{*T}h(x) + \beta/2 \sum_{i=1}^{m} (h_i(x))^2. \qquad (10)$$

where $\beta > 0$. Of course the problem is that λ* is not known at the set of the problem.

The next result suggest an alternative strategy consisting of solving a sequence of problems of the form

$$\text{Minimize } f(x) + \lambda^T_k h(x) + \beta_k/2 \sum_{i=1}^{m} (h_i(x))^2, \qquad \text{where } \lambda_k \in R_{m\times 1}.$$

II. THEOREMS

Theorem 1: In problem (1) assume that f is continuous over F, and that each g_i is continuous over $R_{n\times i}$. Assume also that F is nonempty, closed and both; that each boundary point of positive numbers for which $\lim_{k \to \infty} \beta_k = \infty$ and that for each k there exists an x_k which solves the problem.

$$\text{Minimize } \beta_k \text{ subject to each } g_i(x) < 0 \text{ where}$$

$B_k(x) = f(x) + 1/\beta_k\, b(x)$, and b(x) has the form (2). Then

$$\text{Min}\{ f(x) : g_i(x) \le 0 \text{ for all } i\} = \lim_{k \to \infty} \beta_k(x).$$

Moreover, if is a limit point of any converging subsequence of $\{x_k\}$, then solves problem (1). Finally,

$$\lim_{k \to \infty} (1/\beta_k)\, b(x_k) = 0.$$

The result suggests an interactive procedure for solving (1) namely, select or determine a method for generating an increasing sequence of positive number { β_k } that tends to infinity. Then for each k, solve the problem.

$$\text{Minimize } f(x) + 1/\beta_k\, b(x) \quad \text{subject to } g_i(x) < 0 \text{ for } i = 1, 2, \ldots, m \tag{11}$$

Denote the solution by x_k. The values $f(x_k) + 1/\beta_k\, b(x)$ will usually approach the minimum value of f. The process terminates when the desired accuracy is reached.

Since the process works in the interior of F, it is necessary to determine an initial point x_0 for which each $g_i(x_0) < 0$. The iterative procedure for minimizing $f(x) + 1/\beta_1\, b(x)$ would start at x_0. Finding such a point can itself be difficult, and it is a serious drawback of the method. As with penalty functions, barrier function methods can be slow to converge as the bounding is approached.

Ideally, the barrier function prevents the search from leaving the interior of F by becoming infinite as the boundary of F is approached. Hence in the ideal situation one would not have to worry about any constraint, $g_i(x) < 0$, of (7). However, in practice a line search might step over the boundary into the exterior of F. When this happens it is possible to experience a decrease in b(x), and hence, B(x). For instance, in the example above let $x_k = 5.01$, $\Delta x = 0.02$, $\beta = 1/2$, and hence, $x_{k+1} = x_k - \Delta x = 4.99$. Then $B(5.01) = 205.01$ and $B(4.99) = 195.01$, where $B(x) = x - 2(5 - x)^{-1}$. In general, one must determine if $g_i(x_k) < 0$ for $i = 1, 2, \ldots, m$ for each x_k that is generated.

Theorem2. In problem (7) assume that the functions $h_1, h_2, \ldots, h_m$ and f are all continuous and that the set F of feasible solutions is nonempty, closed and bounded. Let $\{\lambda_k\}$be a sequence of bounded vectors in $R_{m\times 1}$, { β_k } be a sequence of increasing positive numbers that tends to infinity, and let X_k solve the problem:

$$\text{Minimize } L_k(x, \lambda_k) = f(x) + \lambda^T_k\, h(x) + \beta_k/2\, p(x), \tag{12}$$

Where $p(x) = \sum_{i=1}^{m} (h_i(x))^2$. Then every accumulation point of $\{x_k\}$ solves (7). In particular if a sequence $\{x_{ki}\}$ of $\{x_k\}$ converges to $\bar{x}$, then

(1) $p(\bar{x}) = \lim_{k_i \to \infty} p(x_{ki}) = 0$

(2) $\lim_{k_i \to \infty} \beta_{ki} / 2\, p(x_i) = 0$

(3) $\min (f(x) : x \in F\} = \lim_{k_i \to \infty} L_{ki}(x, \lambda_{ki})$.

Proof: Suppose that the sequence $\{x_{ki}\}$ of $\{x_k\}$ converges to $\bar{x}$. Since corresponding subsequence $\{\lambda_{ki}\}$ of $\{\lambda_k\}$ is bounded it possess accumulation points. Let $\bar{\lambda}$ be one of these accumulation points. Then there is a subsequence of $\{\lambda_{ki}\}$ that convergence to $\bar{\lambda}$. The corresponding subsequences of $\{x_{ki}\}$ still converges to $\bar{x}$. To simplify notation, we shall denote both of these convergence subsequences by $\{\lambda_{ki}\}$ and $\{x_{ki}\}$, respectively. Thus $\lambda_{ki} \to \bar{\lambda}$ and $x_{ki} \to \bar{x}$ as $k_i \to \infty$.

Let x* denote a solution of problem (7) from the definition of x_{ki}, it follows that

$$L_{ki}(x_{ki}, \lambda_{ki}) \le L_{ki}(x, \lambda_{ki})$$

for all $x \in R_{n\times 1}$. Since $h(x) = 0$ and $p(x) = 0$ when $x \in F$, it follows from this least inequality that

$$\begin{aligned} L_{ki}(x_{ki}, \lambda_{ki}) &\le \inf \{ L_{ki}(x, \lambda_{ki}) : x \in R_{n\times 1} \} \\ &\le \inf \{ L_{ki}(x, \lambda_{ki}) : x \in F \} \\ &= \inf \{ f(x) : x \in F \} \\ &= f(x^*) \end{aligned}$$

Hence, $0 \le \beta_{ki}/2\, p(x_{ki}) \le f(x^*) - f(x_{ki}) - \lambda^T_{ki}\, h(x_{ki})$ (13)

Since the limit of the far right side of expression (13) exists, it follows that

$$M = \text{Sup} \left\{ \beta_{ki}/2\, p(x_{ki}) \right\}$$

exists. This derives $p(\bar{x}) = \lim_{k_i \to \infty} p(x_{ki}) = 0$. Thus, $h(\bar{x}) = 0$ and $\bar{x} \in F$, so that $f(x^*) \le f(\bar{x})$. Since the sequence $\bar{x}$ $\{\lambda_{ki}\}$ is bounded, it follows from (13) that $0 \le f(x^*) \le f(\bar{x})$. Thus, $f(\bar{x}) \le f(x^*)$, therefore,

$f(x^*) = f(\bar{x})$ and $M = 0$. It can be shown that

$$f(\bar{x}) = \lim_{k_i \to \infty} L_k(x_{ki}, \lambda_{ki}).$$

This completes the proof of the theorem.

The above proof gives no indication of how the sequence $\{\lambda_k\}$ could possibly be generated. We suggest one approach now.

Suppose that $h_1, h_2, \ldots, h_m$ and f are twice continuously differentiable. Let $\{\beta_k\}$ be an increasing sequence of positive numbers that tends to infinity, and let $\{\varepsilon_k\}$ be a decreasing sequence of nonnegative numbers that tends to 0. Select a vector $\lambda_1 \in R_{m\times 1}$ and determine a solution x_1 of the problem.

$$\text{Minimize } L_1(x, \lambda_1) = f(x) + \lambda^T_1 h(x) + \beta_1/2\, p(x),$$

$$\text{such that } \left\| \frac{\partial}{\partial x} L_1(x_1, \lambda_1) \right\| \le \varepsilon_1.$$

Then define λ_2 by the expression

$$\lambda_2 = \lambda_1 + \beta_1 h(x_1).$$

Next determine the solution x_2 of the problem

$$\text{Minimize } L_2(x, \lambda_2) = f(x) + \lambda^T_2 h(x) + \beta_2/2\, p(x)$$

$$\text{such that } \left\| \frac{\partial}{\partial x} L_2(x_2, \lambda_2) \right\| \le \varepsilon_2.$$

Continuing in this fashion, two sequences $\{\lambda_k\}$ and $\{x_k\}$ are generated so that

$$\lambda_{k+1} = \lambda_k + \beta_k h(x_k) \tag{14}$$

and

$$\left\| \frac{\partial}{\partial x} L_k(x_k, \lambda_k) \right\| \le \varepsilon_k.$$

Note that

$$\left[\frac{\partial}{\partial x} L_k(x_k, \lambda_k) \right]^T = \nabla f(x_k)^T + \lambda^T_k \frac{\partial h}{\partial x}(x_k) + \beta_k h(x_k)^T \frac{\partial h}{\partial x}(x_k)$$

$$= \nabla f(x_k)^T + (\lambda^T_k + \beta_k h(x_k)^T) \frac{\partial h}{\partial x}(x_k) \tag{15}$$

Now suppose that a subsequence $\{x_{ki}\}$ of $\{x_k\}$ converges to a vector $\bar{x}$. Assume that $\left| \frac{\partial h}{\partial x}(\bar{x}) \right| \ne 0$. Then when x_{ki} is near $\bar{x}$, it follows that $\left| \frac{\partial h}{\partial x}(x_k) \right| \ne 0$ and $\frac{\partial h}{\partial x}(x_k)$ is nonsingular. It follows from (15) that

$$\left[\frac{\partial}{\partial x} L_{ki}(x_{ki}, \lambda_{ki}) \right]^T \left[\frac{\partial h}{\partial x}(x_{ki}) \right]^{-1} = \nabla f(x_{ki})^T \left[\frac{\partial h}{\partial x}(xki)^{-1} \right] + (\lambda^T_{ki} + \beta_{ki} h(x_{ki})^T). \tag{16}$$

But

$$\left[\frac{\partial}{\partial x} L_{ki}(x_{ki}, \lambda_{ki}) \right]^T \to 0 \text{ as } k_i \to \infty.$$

So, the left side of (16) tends to zero as $k_i \to \infty$. It follows that

$$(\lambda^T_{ki} + \beta_{ki} h(x_{ki})^T) \to \bar{\lambda},$$

where $\bar{\lambda} = - \nabla f(\bar{x})^T \left(\frac{\partial h}{\partial x} (\bar{x}) \right)^{-1}$. (17)

when the sequence $\{\lambda_{ki}\}$ is bound, then it can be shown that $h(\bar{x}) = 0$ and $\bar{x}$ and solves (7). Note that

$$\nabla f(\bar{x})^T + \bar{\lambda}^T \frac{\partial h}{\partial x} (\bar{x}) = 0.$$

Hence, $\bar{x}$ and $\bar{\lambda}$ satisfy equations (8). This suggests the following iterative procedure for solving the problem (7).Select an increasing sequence $\{\beta_k\}$ of positive numbers that tends to infinity, and a sequence $\{\varepsilon_k\}$ of nonnegative numbers that tends to zero. Let k =1. Let us put the problem in the form of an algorithm2.

III. ALGORITHMS

Algorithm1:

Step 1: Determine a point x_0 that satisfies the inequalities $g_i(x) < 0$ for i = 1, 2, …., m.

Step 2: Determine a sequence of increasing positive number $\{\beta_k\}$ that tends to infinity.
Let k = 1.

Step 3: Determine a solution x_k of the problem.

Minimize $f(x) - 1/\beta_k \sum_{i=1}^{m} 1/g_i(x)$ subject to $g_i(x) < 0$ for i = 1, 2, ….,m.

Check to be sure that $g_i(x_k) < 0$ for each i before continuing. Start the search with x_{k-1}.

Step 4: If the required accuracy has been achieved, then stop. Otherwise replace k with k + 1 and return to step 3.

Algorithm2:

Step 1: Determine a solution x_k of the problem:
Minimize $f(x) + \lambda^T{}_k(x) + \beta_k / 2\, p(x)$
that satisfies

$$\left\| \frac{\partial}{\partial x} L_k(x_k, \lambda_k) \right\| \le \varepsilon_k.$$

If the required accuracy has been achieved, then stop. Otherwise go to step2.

Step 2: Compute $\lambda_{k+1} = \lambda_k + \beta_k h(x_k)$. Replace k by k + 1 and return to step 1.

IV. EXAMPLES

Example1 : Use barrier function method to solve-

$$\text{Minimize } B_k(x) = f(x)\ 1/\beta_k\, b(x),$$

where $b(x) = -(x^2{}_1 + x^2{}_2 - 1)^{-1}$, $\beta_1 = 1$ and $\beta_k = 2\beta_{k-1}$. The results are summarized in Table1.

Table1: Barrier Function Method

K	B_K	x_k	$B_k(x_k)$	$f(x_k)$	$B(x_k)$
1	1	$[-0.31810\ \ -0.31810]^T$	-0.61752	-0.63620	1.25372
2	2	$[-0.41964\ \ -0.41964]^T$	-0.06744	-0.83928	1.54368
3	4	$[-0.50000\ \ -0.50000]^T$	-0.50000	-1.00000	2.00000
4	8	$[-0.55947\ \ -0.55947]^T$	-0.78470	-1.11894	2.67389
5	16	$[-0.60233\ \ -0.60233]^T$	-0.97689	-1.20466	3.64435
.	.	.	.	.	.
.	.	.	.	.	.
20	524288	$[-0.70629\ \ -0.70629]^T$	-1.41175	-1.41258	433.11043
21	1048576	$[-0.70653\ \ -0.70653]^T$	-1.41248	-1.41306	613.22606
.	.	.	.	.	.
.	.	.	.	.	.
36	2^{35}	$[-0.70711\ \ -0.70711]^T$	-1.41422	-1.41422	109890.11

Min $f(x) = x_1 + x_2$ subject to $x^2{}_1 + x^2{}_2 \le 1$

Example2: Let us apply the Quadratic Penalty Function Method to solve –

$$\text{Minimize } f(x) = x_1 + x_2 \text{ subject to } x^2_1 + x^2_2 \le 1.$$

Solution: Rewrite the above problem as

$$\text{Minimize } f(x) = x_1 + x_2 \text{ subject to } 1 - x^2_1 - x^2_2 - x^2_3 = 0.$$

Next set $\lambda_1 = 1$, $\beta_1 = 1$, $\varepsilon_1 = 0.001$, $\beta_{k+1} = 4\beta_k$, $\lambda_{k+1} = \lambda_k + \beta_k h(x_k)$, where $h(x) = 1 - x^2_1 - x^2_2 - x^2_3$ and $\varepsilon_{k+1} = \varepsilon_k / 10$. Then solve the sequence of problems.

$$\text{Minimize } L_k(x, \lambda_k) = f(x) + \lambda_k(1 - x^2_1 - x^2_3) + \beta_k / 2\,(1 - x^2_1 - x^2_2 - x^2_3)^2.$$

Using Newton's method in the direction of steepest descent beginning with the point $[\,2\ \ 0\ \ 0\,]^T$ where k = 1. The results of calculation are given in Table2.

Table2: Quadratic Penalty Function Method

K	β_K	λ_K	x_K	$f(x_K)$	ε_k	$\left\Vert \frac{\partial L}{\partial x}(x_K, \lambda_K) \right\Vert$
1	1	1	$[-1.10776\ \ -1.10654\ \ 0]^T$	-2.21430	10^{-3}	7.933×10^{-4}
2	4	-0.4516	$[-0.72766\ \ -0.72761\ \ 0]^T$	-1.45527	10^{-4}	4.817×10^{-5}
3	16	-0.6872	$[-0.70754\ \ -0.70754\ \ 0]^T$	-1.41508	10^{-5}	1.615×10^{-6}
4	64	-0.7067	$[-0.70711\ \ -0.70711\ \ 0]^T$	-1.41422	10^{-6}	1.267×10^{-7}
5	256	-0.7071	$[-0.70711\ \ -0.70711\ \ 0]^T$	-1.41422	10^{-7}	5.508×10^{-10}

$$\text{Min } f(x) = x_1 + x_2 \text{ subject to } x^2_1 - x^2_2 \le 1$$

V. CONCLUSION

Barrier function method and Quadratic Penalty Function Method are added to the objective function and the resulting function is minimized. The difference is that the solutions are interior points. The purpose of the Barrier function and the Quadratic Penalty Function Method is to prevent to the solutions from leaving the interior point and we also derived algorithms. Using Barrier function method and Quadratic Penalty Function Method, we have solved the examples and the results that are summarized in the Tables. Thus we show that the Constrained Problem is converted into Unconstrained Problem by Barrier Function Method and Quadratic Penalty Function Method.

REFERENCES

[1.] Gupta Prem Kumar and Dr. D. S. Hira, Operation Research, S. Chand & Company, 2012.

[2.] Sharma, S. D., Operations Research, Merrut Publication, 2009.

[3.] Swarup, K., Operation Research, S. Chand & Sons, New Delhi, 2010.

[4.] Taha H.A., Operation Research, Prentice Hall, 2002.

[5.] Hillier F. and Lieberman G. , Introduction to Operation Research Holden-Day, 2001.

[6.] Wallace, S.W., Algorithms and Model formulations in Mathematical Programming, 1989.

[7.] Bertsekas, D.P., Constrained Optimization and Lagrange Multiplier Methods, Academic Press, New York, 1990.

[8.] Conn, A.R., Penalty Function Methods in Nonlinear Optimization, Academic Press, New York, 1985.

[9.] Fletcher, R., Methods for Nonlinear Constraints, In Nonlinear Optimization, Academic Press, New York, 2003.

[10.] Powell, M.J.D., A Fast Algorithm for Nonlinearly Constrained Optimization Calculations, in Lecture Notes in Mathematics 630, G. A. Watson, Springer-Verlag, Berlin, PP. 144-157, 1998.

[11.] Pshenichyni, B.N., Algorithms for the General Problem of Mathematical Programming, Kibernetika 6, PP.120-125, 1995.