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ON SOME ASPECTS OF u -IDEALS DETERMINED BY ℓ_1 AND c_0

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Abstract: *The M -ideals defined on a real Banach space are called u -ideals. The u -ideals containing isomorphic copies of ℓ_1 are not strict u -ideals. In this paper we show that u -ideals with unconditional basis (x_n) which is shrinking has no isomorphic copy of ℓ_1 and thus a strict u -ideal. Finally we show that u -ideals with unconditional basis (x_n) which is boundedly complete is not homeomorphic to copies of c_0 implying that they are weak* closed in their biduals X^{**}.*

Key Words: *u -ideals, strict u -ideals, Shrinking, boundedly complete*

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I. INTRODUCTION

A sequence $(x_n)_{i=1}^{\infty}$ is called a basis of a normed space X if for every $x \in X$ there exists a unique series $\sum_{i \geq 1} a_i x_i$ that converges to x. The basis $(x_n)_{i=1}^{\infty}$ for a Banach space X is an unconditional basis if for each $x \in X$ there exists a unique expansion of the form $x = \sum_{n=1}^{\infty} a_n x_n$ where the sum converges unconditionally. The basis (x_n) is said to be boundedly complete whenever given a sequence (a_n) of scalars for which $\left\{\sum_{k=1}^{n} a_k x_k : n \geq 1\right\}$ is bounded, then $\lim_n \sum_{k=1}^{n} a_k x_K$ exists. If $x_n \in X$ is not boundedly complete then X contains an isomorphic copy of c_0 [1].

Let $y \in X^{**}$ which belongs to X. We say that (x_n) is shrinking,

$$\left(\left\langle u, \sum_{i=1}^{n} \langle y, u_i \rangle . x_i \right\rangle\right)_n = \left(\left\langle y, \sum_{i=1}^{n} \langle u, x_i \rangle . u_i \right\rangle\right)_n \text{ converges to } \langle y, u \rangle \text{ for every } u \in X^*.$$

The notion of u -ideals was introduced and studied thoroughly by Godfrey et al [2]. They generalized M -ideals defined on a real Banach space. In their paper on unconditional Ideals they established that u -ideals containing copies of ℓ_1 are not strict u -ideals. A Banach space X is said to be a strict u -ideal in its bidual when the canonical decomposition $X^{***} = X^* \oplus X^{\perp}$ is unconditional. In other words for X to be a strict u -ideal the u -complement of $X^{\perp}$ must be norming, that is, the range V of the induced projection on X^{***} is a norming subspace of X^*. Vegard and Asvald $[3]$ characterized Banach spaces which are strict u -ideals in their biduals and showed that X is a strict u -ideal in a Banach space Y if it contains c_0. Matuya et al [4] using the approximation properties, hermitian conditions, isometry studied properties of u -ideals and their characterization. They showed that u -ideals containing no copies of sequence space ℓ_1 are strict u -ideals. In this paper we show that Banach spaces with unconditional basis (x_n) that is shrinking is not homeomorphic to copies of ℓ_1 and so it a strict u -ideal. We also show that the u -ideals with unconditional basis which are boundedly complete are not bicontinuous to c_0 meaning their u -complement is weak* closed.

II. RESULTS ON STRICT u -IDEALS

2.1 Proposition

Let X be u -ideal with unconditional basis (x_n). The following statements are equivalent:

[1] The sequence (x_n) is shrinking

[2] X contains no copy of ℓ_1

[3] X is a strict u -ideal

Proof: $i \Rightarrow ii$ it is clear from the definition that if a sequence (x_n) is shrinking then X contains no isomorphic copy of ℓ_1. In fact proving by contradiction it suffices to show that X contains a copy of ℓ_1. Let (c_n) be a sequence of coefficient functional associated with (x_n). Our hypothesis applies that, for some $c \in X^{**}$, the series $\sum_{n=1}^{\infty} \langle c, x_n \rangle . c_n$ does not converge in $\left[X^{**}, \rho\left(X^{**}, X \right) \right]$ implying that $\left(\sum_{n=1}^{k} \langle c, x_n \rangle . c_n \right)_k$ cannot be Cauchy in $\left[X^{**}, \rho\left(X^{**}, X \right) \right]$. We therefore find a bounded set $L \subset \ell_1$ and for each $k \in \mathbb{N}$ we define maps $P, Q : L^{\mathbb{N}} \to X$ by $P(\xi) = \sum_{n=1}^{\infty} \langle c, x_n \rangle . c_n$ and $Q(\xi) = \left(\sum_{n=1}^{k} \langle c, x_n \rangle . c_n \right)_k$ which implies together with $L\left(P\xi - Q\xi \right) \le 2\|\xi\|$ that $L\left(Q\xi \right) \ge L\left(P\xi \right) - 2\|\xi\| \ge (\xi)$ holds. P is continuous since (x_n) is bounded, so that P maps $L^{\mathbb{N}}$ homeomorphically into X. Now $L^{\mathbb{N}}$ is dense in ℓ_1. Since ℓ_1 is weakly sequentially complete the same is true for the subspace $P(\ell_1)$ of X. In particular (c_n) has a weak limit in X and so does (x_n) because of $\langle x, c_k \rangle = \lim_{n \to \infty} \langle x_n, c_k \rangle, \forall\ k \in \mathbb{N}$, this limit has to be x and this is the desired contradiction.

$ii \Rightarrow iii$ Considering the map P it is clear from the definition that if $x^* \in X^*$ and $P x^* = x^*$ then $x^* \in U$ where $U = p\left(x^{***} \right)$. Now for each $\eta \in X^{**}$ we consider the set $F_\eta = \left\{ x^* \in X \mid P\eta\left(x^* \right) = \eta x^* \right\}$ then there is a net $\left(x_d^* \right) \in B_{X^*}$ converging in the weak*-topology of X^{***} to u. However, $u - x^* \in X^{\perp}$ so that x_d^* converges to x^*. Thus $u(\eta) = \lim_d \eta\left(x_d^* \right) = \eta\left(x^* \right)$ so that $u x^*(\eta) = \eta\left(x^* \right)$. Now $P\eta\left(x^* \right) = u x^*(\eta)$ so we conclude that $x^* \in F_\eta$. Hence F_η is norming.

$iii \Rightarrow i$ Suppose X is strict u -ideal. We proceed to show that (x_n) is shrinking. Let $y \in X^{**}$ which belongs to X. Since (x_n) is shrinking, $\left(\left\langle u, \sum_{i=1}^{n} \langle y, u_i \rangle . x_i \right\rangle \right)_n = \left(\left\langle y, \sum_{i=1}^{n} \langle u, x_i \rangle . u_i \right\rangle \right)_n$

converges to $\langle y,u\rangle$ for every $u\in X^{*}$. We conclude from this that $\left(\sum_{i=1}^{n}\langle y,u_i\rangle .x_i\right)$ is bounded in X so it has a limit say

x . $\langle x,u\rangle=\sum_{i=1}^{\infty}\langle y,u_i\rangle.\langle x_i,u\rangle=\langle y,u\rangle$, for all $u\in X^{*}$,

we get $y=x\in X$.

2.2 Proposition Let X be u **-**ideal with unconditional basis (x_n). Show that if X contains no copy of c_0 then

[1] (x_n) is boundedly complete

[2] (x_n^{*}) is weak* closed in X^{**}

Suppose (x_n) is not boundedly complete. Then there exists $(\lambda_i)\in T$ with $\left(\sum_{i=1}^{n}\lambda_i x_i\right)_n$ bounded but not Cauchy in X. Let now $e\in\Psi$ and let (m_i) and (n_i) be increasing sequences in $\mathbb{N}$ such that $m_i<n_i<m_{i+1}$ and $a_i=\sum_{m_i}\lambda_j x_j\notin u_e$, that is, $e(a_i)>1,\ \forall\ i\in\mathbb{N}$. Here we set $M_i=\{m_i,m_{i+k},\ldots\ldots,n_i\}$. Choose $u_i\in U_e^{0}$ such that $\left|\langle u_i,a_i\rangle\right|>1$, $\forall\ i\in\mathbb{N}$ and let $f\in\Psi$ be such that

$$e\left(e_{M,\delta}x\right)\le f(x),\ \forall\ x\in X\ ,\ M\in\mathbb{N}\text{ and }\delta\in\Delta\ .$$

By construction of the sets M_i, it follows that

$$g\left(\sum_{i\in\mathbb{N}}a_i\right)=g\left(\sum_{i\in\mathbb{N}}\sum_{j\in M_i}\lambda_i x_i\right)\le e_g\ ,$$

Whence $\left|\left\langle v,\sum_{i\in\mathbb{N}}a_i\right\rangle\right|\le e_g\quad\forall\ g\in\Psi\ ,\ v\in U_g^{0}$.

There exists a constant $c>0$ such that

$$\sum_{i\in\mathbb{N}}\left|\langle v,a_i\rangle\right|\le c.\,e_g\quad\forall\ g\in\Psi\ ,\ v\in U_g^{0}.$$

Consider now T as a subspace of c_0 and define

$\beta:T\to X$ by $\beta(\eta)=\sum_{n}\eta_n a_n$.

Then β is linear and continuous, because of

$$g(\beta\eta)=g\left(\sum_{n}\eta_n a_n\right)$$

$$=\operatorname{lub}_{U_g^{0}}\left|\left\langle v,\sum_{n}\eta_n a_n\right\rangle\right|\le c.e_g.\|\eta\|_{\infty},\ \forall\ g\in\Psi\ ,\eta\in T\ .$$

β is also open , because given $\eta\in T$ and $j\in\mathbb{N}$ such that $\left|\eta_j\right|=\|\eta\|_{\infty}$, then we can choose $\delta\in\Delta$ such that

$\delta_i \eta_i \langle u_j, a_i \rangle = |\eta_i \langle u_j, a_i \rangle|$ for all $i \in \mathbb{N}$.

Therefore $\|\eta\|_\infty = |\eta_j| \le \sum_n |\eta_n| |\langle u_j, a_n \rangle|$

$$= \left\langle v_j, \sum_n \delta_n \eta_n a_n \right\rangle \le e\left(\sum_n \delta_n \eta_n a_n \right)$$

$$\le f(\beta \eta) .$$

Thus β is bicontinuous between T and X . This implies that T is dense in c_0 and since X is sequencially complete, β extends a linear homeomorphic embedding c_0 into X which is a contradiction.

We shall show that when $(x_n^*) \subset X^*$ and $(x_n^{**}) \subset X^{**}$ satisfy

$$\sum_{n=1}^{\infty} \|x_n^*\| \|x_n^{**}\| < \infty$$

and

$$\sum_{n=1}^{\infty} x_n^{**}(s^* x_n^*) = 0 \text{ for } s \in F(X, X),$$

then

$$\sum_{n=1}^{\infty} x_n^{**}(x_n^*) = 0 .$$

We may assume that

$$\|x_n^*\| \to 0 \text{ and } M = \sum_{n=1}^{\infty} \|x^{**}\| < \infty .$$

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\sum_{n>N} \|x^{**}\| < \frac{\varepsilon}{4}$. Since X has the weak approximating sequence, for $\lambda = {L^*}/_{\hat{X}} \in K(\hat{X}, Z^*)$, there exists a net $(s_\alpha) \subset F(\hat{X}, \hat{X})$ such that $\text{lub}_\alpha \|\lambda s_\alpha\| \le \|\lambda\| \le 1$

and

$$(L^* s_\alpha x)(k) \to (L^* x)(k) = (Lk)(x) .$$

Therefore $s_\alpha^* x^* \to x^*$ weak* for all $x^* \in L(k)$. In particular $s_\alpha^* x^* \to x^*$ for all $x^* \in F$. Since $\|\beta s_\alpha\| \le 1$, we also have $\|s_\alpha^* L\| \le 1$, if $\|x^*\| = 1$ then $x^* = Lk$ for some $k \in K$. Thus $\|s_\alpha^* x^*\| \le 1$, so $s_\alpha^* \to x^*$ is weak* convergent. Hence there is some s_α such that

$$\|x_n^* - s_\alpha^* x_n^*\| < \frac{\varepsilon}{2M} , \; n = 1, \ldots, N$$

Now we have

$$\left| \sum_{n=1}^{\infty} x_n^{**} \right| = \sum_{n=1}^{\infty} x_n^{**} (x_n^* - s_\alpha^* x_n^*)$$

$$\le \sum_{n=1}^{N} \|x_n^{**}\| \|x_n^* - s_\alpha^* x_n^*\| + \sum_{n>N} \|x_n^{**}\| \|L k_n - s_\alpha^* L k_n\|$$

$$\le \frac{\varepsilon}{2} + \sum_{n>N} \|x_n^{**}\| (\|L\| + \|s_\alpha^* L\|) \|k_n\|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence

$$\sum_{n=1}^{\infty} x_n^{**}\left(x_n^{*}\right) = 0$$

III. CONCLUSION

We have shown that u -ideals can be characterized using sequence spaces. In particular we considered the sequence spaces ℓ_1 and c_0. The u -ideals containing no isomorphic copies of ℓ_1 are strict u -ideals whereas those that are not homeomorphic to the copies of c_0 their u -complement is weak* -closed

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