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On The Zeros of Analytic Functions

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Abstract : *In this paper we consider a certain class of analytic functions whose coefficients are restricted to certain conditions, and find some interesting zero-free regions for them. Our results generalize a number of already known results in this direction.*

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I. INTRODUCTION AND STATEMENT OF RESULTS

Regarding the zeros of a class of analytic functions , whose coefficients are restricted to certain conditions, W. M. Shah and A. Liman [4] have proved the following results:

Theorem A: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic in $|z|<t$. If for some $k \geq 1$,

$$k|a_0| \geq t|a_1| \geq t^2|a_2| \geq ......,$$

and for some β ,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2},\ j=0,1,2,......,$$

then f(z) does not vanish in

$$\left| z - \frac{(k-1)t}{M^2-(k-1)^2} \right| \leq \frac{Mt}{M^2-(k-1)^2} ,$$

where

$$M = k(\cos\alpha + \sin\alpha) + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j .$$

Theorem B: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic in $|z| \leq t$. If for some $k \geq 1$ and $\lambda > 0$,

$$k|a_0| \leq t|a_1| \leq t^2 a_2 \leq ...... \leq t^k|a_k| \geq t^{k+1} \geq ...... ,$$

and for some β ,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2},\ j=0,1,2,......,$$

then f(z) does not vanish in

$$\left| z - \frac{(k-1)t}{M*^2-(k-1)^2} \right| \leq \frac{M^* t}{M^{*2}-(k-1)^2} ,$$

where

$$M^* = (\frac{2|a_k|}{|a_0|}t^k - k)\cos\alpha + k\sin\alpha + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j .$$

The aim of this paper is to generalize the above - mentioned results. In fact , we are going to prove the following results:

Theorem 1: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic for $|z|\le t$. If for some $\rho \ge 0$,

$\rho + |a_0| \ge t|a_1| \ge t^2|a_2| \ge ....,$

and for some real β and α,

$\left|\arg a_j - \beta\right| \le \alpha \le \frac{\pi}{2},\ j=0,1,2,......,$

then f(z) does not vanish in

$$\left| z - \frac{\rho|a_0|t}{|a_0|^2 M^2 - \rho^2} \right| < \frac{Mt|a_0|^2}{|a_0|^2 M^2 - \rho^2},$$

where

$$M = \left(\frac{\rho + |a_0|}{|a_0|}\right)(\cos\alpha + \sin\alpha) + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j .$$

Remark1: Taking $\rho=(k-1)|a_0|$, $k\ge 1$, Theorem 1 reduces to Theorem A.

Also taking $\alpha=\beta=0$, we get the following result, proved earlier by Aziz and Shah [2] :

Corollary 1: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic for $|z|\le t$ such that for some $k\ge 1$,

$ka_0 \ge ta_1 \ge t^2 a_2 \ge ......$.

Then f(z) does not vanish in

$$\left| z - \frac{(k-1)t}{2k-1} \right| < \frac{kt}{2k-1}.$$

Taking $\alpha=\beta=0$ and $\rho=0$, we get the following result proved earlier by Aziz and Mohammad [1] :

Corollary 2: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic for $|z|\le t$ such that $a_j>0$ and

$a_{j-1} - ta_j \ge 0,\ j=1,2,3,.......$ Then f(z) does not vanish in $|z|<t$.

Theorem 2: Let $f(z)=\sum_{j=0}^{\infty} a_j z^j \neq 0$ be analytic for $|z|\le t$. If for some $\rho \ge 0$,

$\lambda > 0$,

$|\rho + a_0| \le t|a_1| \le t^2|a_2| \le ...... \le t^{\lambda}|a_{\lambda}| \ge t^{\lambda+1}|a_{\lambda+1}| \ge ....,$

and for some real β and α ,

$\left|\arg a_j - \beta\right| \le \alpha \le \frac{\pi}{2},\ j=0,1,2,......,$

then f(z) does not vanish in

$$\left| z - \frac{\rho|a_0|t}{|a_0|^2 M^{*2} - \rho^2} \right| < \frac{M^{*}t|a_0|^2}{|a_0|^2 M^{*2} - \rho^2},$$

where

$$M^{*} = \left[\left(2\left|\frac{a\lambda}{a_0}\right|t^{\lambda} - \left|1+\frac{\rho}{a_0}\right|\right)\cos\alpha + \left|1+\frac{\rho}{a_0}\right|\sin\alpha + \frac{2\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j \right].$$

Remark2: Taking $\rho=(k-1)|a_0|$, $k\ge 1$, Theorem 2 reduces to Theorem B.

The result of Aziz and Mohammad (Theorem 6 of [1]) follows from Theorem 2 by taking $\rho=0$.

II. LEMMA

For the proofs of the above theorems , we need the following lemma, which is due to Govil and Rahman [3]:

Lemma : If $\left|\arg a_j - \beta\right| \le \alpha \le \dfrac{\pi}{2}$ and for some t>0, $\left|ta_j\right| \ge \left|a_{j-1}\right|, j = 0,1,......,n,$, then $\left|ta_j - a_{j-1}\right| \le [(\left|ta_j\right| - \left|a_{j-1}\right|)\cos\alpha + (\left|ta_j\right| + \left|a_{j-1}\right|)\sin\alpha]$.

III. Proofs Of The Theorems

Proof of Theorem 1: Consider the function

$$F(z) = (z-t)f(z)$$
$$= (z-t)(a_0 + a_1 z + a_2 z^2 + ...)$$
$$= -ta_0 + (a_0 - ta_1)z + (a_1 - ta_2)z^2 + ...$$
$$= -ta_0 - \rho z + (\rho + a_0 - ta_1)z + (a_1 - ta_2)z^2 + ...$$
$$= -ta_0 - \rho z + G(z) \ ,$$

where

$$G(z) = (\rho + a_0 - ta_1)z + \sum_{j=2}^{\infty}(a_{j-1} - ta_j)z^j \ .$$

Since $\left|\arg a_j - \beta\right| \le \alpha \le \dfrac{\pi}{2}, j = 0,1,2,......,$ by using the lemma and the hypothesis, we have for $|z| = t$,

$$|G(z)| \le t[\{(\rho + |a_0|) - |ta_1|\}\cos\alpha + \{(\rho + |a_0|) + |ta_1|\}\sin\alpha$$
$$+ t\{(|a_1| - |ta_2|)\cos\alpha + (|a_1| + |ta_2|)\sin\alpha\} + ...]$$
$$\le t|a_0|[(1 + \frac{\rho}{|a_0|})(\cos\alpha + \sin\alpha) + \frac{2\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j]$$
$$= t|a_0|M.$$

Since G(z) is analytic for $|z| \le t$, G(0)=0, it follows by Schwarz's lemma that

$$|G(z)| \le t|a_0|M|z| \quad \text{for } |z| \le t \ .$$

Hence it follows that

$$|F(z)| \ge |ta_0 + \rho z| - |G(z)|$$
$$\ge |a_0|[\left|\frac{\rho z}{a_0} + t\right| - t|z|M]$$
$$> 0$$

if

$$\left|\frac{\rho z}{a_0} + t\right| > t|z|M$$

i.e. if

$$t|z|M < \left|\frac{\rho z}{a_0} + t\right| .$$

Since the region defined by

$$t|z|M < \left|\frac{\rho z}{a_0} + t\right|$$

is precisely the disk

$$\left|z-\frac{\rho|a_0|t}{|a_0|^2M^2-\rho^2}\right|<\frac{Mt}{|a_0|^2M^2-\rho^2},$$

we conclude that F(z) and therefore f(z) does not vanish in the disk

$$\left|z-\frac{\rho|a_0|t}{|a_0|^2M^2-\rho^2}\right|<\frac{Mt}{|a_0|^2M^2-\rho^2}.$$

That completes the proof of Theorem 1.

Proof of Theorem 2: Consider the function

$$\begin{aligned}
F(z) &= (z-t)f(z)\\
&= (z-t)(a_0+a_1z+a_2z^2+\ldots\ldots)\\
&= -ta_0+(a_0-ta_1)z+(a_1-ta_2)z^2+\ldots\ldots+(a_{n-1}-ta_n)z^n+\ldots\ldots\\
&= -ta_0-\rho z+(\rho+a_0-ta_1)z+(a_1-ta_2)z^2+\ldots\ldots+(a_k-ta_{k+1})z^{k+1}\\
&\quad+\ldots\ldots+(a_{n-1}-a_n)z^n+\ldots\ldots\\
&= -ta_0-\rho z+G(z)\ .
\end{aligned}$$

Since $\left|\arg a_j-\beta\right|\le\alpha\le\frac{\pi}{2},\ j=0,1,2,\ldots\ldots,$ by using the lemma and the hypothesis, we have for $|z|=t$,

$$\begin{aligned}
|G(z)| &= |(\rho+a_0-ta_1)z+(a_1-ta_2)z^2+\ldots\ldots+(a_\lambda-ta_{\lambda+1})z^{\lambda+1}\\
&\quad+\ldots\ldots+(a_{n-1}-a_n)z^{n+1}+\ldots\ldots\ |\\
&\le t[(|ta_1|-|\rho+a_0|)\cos\alpha+(|ta_1|+|\rho+a_0|)\sin\alpha\\
&\quad+(|t^2a_2|-|ta_1|)\cos\alpha+(|t^2a_2|+|ta_1|)\sin\alpha+\ldots\ldots\\
&\quad+(|t^\lambda a_\lambda|-|t^{\lambda-1}a_{\lambda-1}|)\cos\alpha+(|t^\lambda a_\lambda|+|t^{\lambda-1}a_{\lambda-1}|)\sin\alpha\\
&\quad+(|t^\lambda a_\lambda|-|t^{\lambda+1}a_{\lambda+1}|)\cos\alpha+(|t^\lambda a_\lambda|+|t^{\lambda+1}a_{\lambda+1}|)\sin\alpha\\
&\quad+\ldots\ldots+(|t^{n-1}a_{n-1}|-|t^na_n|)\cos\alpha+(|t^{n-1}a_{n-1}|+|t^na_n|)\sin\alpha\\
&\quad+|t^na_n|+\ldots\ldots]\\
&= t|a_0|\left[(2\left|\frac{a\lambda}{a_0}\right|t^\lambda-\left|1+\frac{\rho}{a_0}\right|)\cos\alpha+\left|1+\frac{\rho}{a_0}\right|\sin\alpha+\frac{2\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j\ \right]\\
&= t|a_0|M^*,
\end{aligned}$$

where

$$M^*=\left[(2\left|\frac{a\lambda}{a_0}\right|t^\lambda-\left|1+\frac{\rho}{a_0}\right|)\cos\alpha+\left|1+\frac{\rho}{a_0}\right|\sin\alpha+\frac{2\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j\ \right].$$

Since G(z) is analytic for $|z|\le t$, G(0)=0, it follows by Schwarz's lemma that

$$|G(z)|\le t|a_0||M^*|z|\quad\text{for}\quad|z|\le t.$$

Hence it follows that

$$\begin{aligned}
|F(z)| &\ge |ta_0+\rho z|-|G(z)|\\
&\ge |a_0|[\left|\frac{\rho z}{a_0}+t\right|-t|z|M^*]\\
&> 0
\end{aligned}$$

if

$$\left|\frac{\rho z}{a_0}+t\right| > t|z|M^*$$

i.e. if

$$t|z|M^* < \left|\frac{\rho z}{a_0}+t\right|.$$

Since the region defined by

$$t|z|M^* < \left|\frac{\rho z}{a_0}+t\right|$$

is precisely the disk

$$\left|z-\frac{\rho|a_0|t}{|a_0|^2M^{*2}-\rho^2}\right| < \frac{M^*t}{|a_0|^2M^{*2}-\rho^2},$$

we conclude that F(z) and therefore f(z) does not vanish in the disk

$$\left|z-\frac{\rho|a_0|t}{|a_0|^2M^{*2}-\rho^2}\right| < \frac{M^*t}{|a_0|^2M^{*2}-\rho^2}.$$

That completes the proof of Theorem 2.

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