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Zero-Free Regions for Polynomials With Restricted Coefficients

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Abstract : *According to a famous result of Enestrom and Kakeya, if*

$$P(z)=a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$$

is a polynomial of degree n such that

$$0<a_n \le a_{n-1} \le ...... \le a_1 \le a_0 ,$$

then P(z) does not vanish in $|z|<1$. *In this paper we relax the hypothesis of this result in several ways and obtain zero-free regions for polynomials with restricted coefficients and thereby present some interesting generalizations and extensions of the Enestrom-Kakeya Theorem.*

Mathematics Subject Classification: *30C10,30C15*

Key-Words and Phrases: *Zero-free regions, Bounds, Polynomials*

1. Introduction And Statement Of Results

The following elegant result on the distribution of zeros of a polynomial is due to Enestrom and Kakeya [6] :

Theorem A : If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0,$$

then P(z) has all its zeros in $|z| \le 1$.

Applying the above result to the polynomial $z^n P(\frac{1}{z})$, we get the following result :

Theorem B : If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$$0<a_n \le a_{n-1} \le ...... \le a_1 \le a_0 ,$$

then P(z) does not vanish in $|z|>1$.

In the literature [1-5, 7,8], there exist several extensions and generalizations of the Enestrom-Kakeya Theorem . Recently B. A. Zargar [9] proved the following results:

Theorem C: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n . If for some real number $k \ge 1$,

$$0<a_n \le a_{n-1} \le ...... \le a_1 \le k a_0 ,$$

then P(z) does not vanish in the disk

$$|z| < \frac{1}{2k-1} .$$

Theorem D: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n . If for some real number $\rho, 0 \le \rho < a_n$,

$$0<a_n - \rho \le a_{n-1} \le ...... \le a_1 \le a_0 ,$$

then P(z) does not vanish in the disk

$$|z| \le \frac{1}{1+\frac{2\rho}{a_0}}.$$

Theorem E: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n . If for some real number $k \ge 1$,

$$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0,$$

then P(z) does not vanish in

$$|z| < \frac{a_0}{2ka_n - a_0}.$$

Theorem F: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n . If for some real number $\rho \ge 0,$,

$$a_n + \rho \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0,$$

then P(z) does not vanish in the disk

$$|z| \le \frac{a_0}{2(a_n+\rho) - a_0}.$$

In this paper we give generalizations of the above mentioned results. In fact, we prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n . If for some real numbers $k \ge 1$ and $\rho \ge 0,$,

$$a_n - \rho \le a_{n-1} \le ...... \le a_1 \le ka_0,$$

then P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{k(a_0+|a_0|) - |a_0| + 2\rho - a_n + |a_n|}.$$

Remark 1: Taking $0=\rho<a_n$, Theorem 1 reduces to Theorem C and taking k=1 and $0 \le \rho < a_n$,, it reduces to Theorem D.

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n . If for some real numbers $\rho \ge 0$ and $0<\tau \le 1$,

$$a_n + \rho \ge a_{n-1} \ge ...... \ge a_1 \ge \tau a_0,$$

then P(z) does not vanish in

$$|z| < \frac{|a_0|}{2\rho + a_n + |a_n| - \tau(a_0+|a_0|) + |a_0|}.$$

Remark 2: Taking $\tau=1$ and $a_0>0$, Theorem 1 reduces to Theorem F and taking $\tau=1$, $a_0>0$ and $\rho=(k-1)a_n, k \ge 1$, it reduces to Theorem E .

Also taking $\rho=(k-1)a_n, k \ge 1$, we get the following result which reduces to Theorem E by taking $a_0>0$ and $\tau=1$.

Theorem 3: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n . If for some real numbers $k \ge 1, 0<\tau \le 1$,

$$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge \tau a_0,$$

then P(z) does not vanish in the disk

$$|z| < \frac{a_0}{2ka_n + (1-2\tau)a_0}.$$

2.Proofs of the Theorems

Proof of Theorem 1: We have

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0.$$

Let

$$Q(z) = z^n P(\frac{1}{z})$$

and

$$F(z) = (z-1)Q(z).$$

Then

$$\begin{aligned} F(z) &= (z-1)(a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n) \\ &= -a_0 z^{n+1} - [(a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + ... + (a_{n-2} - a_{n-1}) z^2 \\ &\qquad + (a_{n-1} - a_n) z + a_n]. \end{aligned}$$

For $|z| > 1$,

$$\begin{aligned} |F(z)| &\geq |a_0||z|^{n+1} - [|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + ... + |a_{n-1} - a_n||z| + |a_n|] \\ &= |a_0||z|^n [|z| - \frac{1}{|a_0|}\{|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + ... + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\}] \\ &> |a_0||z|^n [|z| - \frac{1}{|a_0|}\{|ka_0 - a_1 - ka_0 + a_0| + |a_1 - a_2| + ... + |a_{n-1} - a_n + \rho - \rho| \\ &\qquad + |a_n|\}] \\ &\geq |a_0||z|^n [|z| - \frac{1}{|a_0|}\{(ka_0 - a_1) + (k-1)|a_0| + (a_1 - a_2) + ... + (a_{n-2} - a_{n-1}) \\ &\qquad + (a_{n-1} - a_n + \rho) + \rho + |a_n|\}] \\ &= |a_0||z|^n [|z| - \frac{1}{|a_0|}\{k(a_0 + |a_0|) - |a_0| - a_n + |a_n| + 2\rho\}] \\ &> 0 \end{aligned}$$

if

$$|z| > \frac{1}{|a_0|}[k(a_0 + |a_0|) - |a_0| - a_n + |a_n| + 2\rho].$$

This shows that all the zeros of F(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}[k(a_0 + |a_0|) - |a_0| - a_n + |a_n| + 2\rho].$$

But those zeros of F(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of F(z) and hence Q(z) lie in

$$|z| \leq \frac{1}{|a_0|}[k(a_0 + |a_0|) - |a_0| - a_n + |a_n| + 2\rho].$$

Since $P(z) = z^n Q(\frac{1}{z})$, it follows, by replacing z by $\frac{1}{z}$, that all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{k(a_0 + |a_0|) - |a_0| - a_n + |a_n| + 2\rho}.$$

Hence P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{k(a_0 + |a_0|) - |a_0| - a_n + |a_n| + 2\rho}.$$

That proves Theorem 1.

Proof of Theorem 2: We have

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0.$$

Let

$$Q(z) = z^n P(\frac{1}{z})$$

and

$$F(z) = (z-1)Q(z).$$

Then

$$F(z) = (z-1)(a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n)$$
$$= -a_0 z^{n+1} - [(a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + ... + (a_{n-2} - a_{n-1}) z^2$$
$$+ (a_{n-1} - a_n) z + a_n].$$

For $|z| > 1$,

$$|F(z)| \geq |a_0||z|^{n+1} - [|a_0 - a_1||z|^n + |a_1 - a_2||z|^{n-1} + ... + |a_{n-1} - a_n||z| + |a_n|]$$
$$= |a_0||z|^n [|z| - \frac{1}{|a_0|}\{|a_0 - a_1| + \frac{|a_1 - a_2|}{|z|} + ... + \frac{|a_{n-1} - a_n|}{|z|^{n-1}} + \frac{|a_n|}{|z|^n}\}]$$
$$> |a_0||z|^n [|z| - \frac{1}{|a_0|}\{|\tau a_0 - a_1 - \tau a_0 + a_0| + |a_1 - a_2| + ... + |a_{n-1} - a_n + \rho - \rho|$$
$$+ |a_n|\}]$$
$$= |a_0||z|^n [|z| - \frac{1}{|a_0|}\{(a_1 - \tau a_0) + (1-\tau)|a_0| + (a_2 - a_1) + ... + (a_n + \rho - a_{n-1})$$
$$+ \rho + |a_n|\}]$$
$$= |a_0||z|^n [|z| - \frac{1}{|a_0|}\{|a_0| - \tau(a_0 + |a_0|) + a_n + |a_n| + 2\rho\}]$$
$$> 0$$

if

$$|z| > \frac{1}{|a_0|}\{|a_0| - \tau(a_0 + |a_0|) + a_n + |a_n| + 2\rho\}.$$

This shows that all the zeros of F(z) whose modulus is greater than 1 lie in the closed disk

$$|z| \leq \frac{1}{|a_0|}\{|a_0| - \tau(a_0 + |a_0|) + a_n + |a_n| + 2\rho\}.$$

But those zeros of F(z) whose modulus is less than or equal to 1 already lie in the above disk. Therefore, it follows that all the zeros of F(z) and hence Q(z) lie in

$$|z| \leq \frac{1}{|a_0|}\{|a_0| - \tau(a_0 + |a_0|) + a_n + |a_n| + 2\rho\}.$$

Since $P(z) = z^n Q(\frac{1}{z})$, it follows, by replacing z by $\frac{1}{z}$, that all the zeros of P(z) lie in

$$|z| \geq \frac{|a_0|}{|a_0| - \tau(a_0 + |a_0|) - a_n + |a_n| + 2\rho}.$$

Hence P(z) does not vanish in the disk

$$|z| < \frac{|a_0|}{|a_0| - \tau(a_0 + |a_0|) - a_n + |a_n| + 2\rho}.$$

That proves Theorem 2.

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