

Analysis of Energy Spectra and Wave Function of Trigonometric Poschl-Teller plus Rosen-Morse Non-Central Potential Using Supersymmetric Quantum Mechanics Approach

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Abstract: The Energy Eigenvalues And Eigenfunction Of Trigonometric Poschl-Teller Plus Rosen Morse Non-Central Potential Are Analysis Using Supersymmetric Quantum Mechanics. Trigonometric Poschl-Teller Plus Rosen-Morse Is A Non-Central Shaped Invariance Potential. Recently Developed Supersymmetric In Field Theory Has Been Successfully Employed To Make A Complete Mathematical Analysis Of The Reason Behind Exact Solvability Of Some Non-Central Potentials In A Close Form. Then, By Operating The Lowering Operator We Get The Ground State Wave Function, And The Excited State Wave Functions Are Obtained By Operating Raising Operator Repeatedly. The Energy Eigenvalue Is Expressed In The Closed Form Obtained Using The Shape Invariant Properties

Keyword: Supersymmetric method, Non-central potentials, Trigonometric Poschl Teller plus Rosen Morse

I. Introduction

One of the important tasks of quantum mechanic is finding accurate exact solution of Schrödinger equation for a certain potential [1][3]. It is obvious that finding exact solution of SE by the usual and traditional methods is impossible for real physical system, except certain cases such as a system with oscillator harmonic and hydrogen atom[2-5]. Thus, it is inevitable to use new methods to help us solve the real physical system. Among the cases where we have to refuse ordinary methods and seek new methods are in solving SE with non-central potentials. Accordingly, different methods are used to solve SE with non-central potentials among which we can name, factorization method [6-9], NU method [8-10] [11,16], supersymmetry (SUSY QM) [11-13], and Romanovsky Polinomials[12] [13]. Supersymmetri is, by definition [14][15], a symmetry between fermions and boson. A supersymmetric field theoretical model consists of a set of quantum fields and of a lagrangian for them which exhibit such a symmetry. The Lagrangian determines, through the action pricipale, the equations of motion and hence the dynamical behaviour of the particle. Supersymmetry theories describe model worlds of particles, created from the vacuum by the fields, and the interactions between these particles. The supersymmetry manifests itself in the particle spectrum and in stringent relationship between different interaction processes even if these involve particles of different spin and of different statistics. Recently, some authors have investigated on solving Schrödinger equation with physical potentials including Poschl-Teller potential [16,17], Non-central potential [18], Hulthén plus Manning-Rosen potential[19], trigonometric Rosen-Morse potential and Scarf potential[20], Eckart potential Using NU method[21], and trigonometric Poschl-Teller potential plus Rosen Morse by using Romanovsky polinomial [22]. In this paper, we investigate the energy eigenvalues and eigenfunction of trigonometric Poschl-Teller potential plus Rosen Morse non-central potentials using SUSYQM method. The trigonometric Poschl-Teller was used to describe molecular vibrations, while the trigonometric Rosen-Morse potential was used to describe the essential of the QCD quark-gluon dynamics in the regime suited of the asymptotical freedom of the quarks [23-25]. The angular wave functions are visualized using Maple 12.

II. Review Of Formula For Supersymmetric Quantum Mechanics

2.1. Supersymmetry Quantum Mechanics (SUSY QM)

Witten defined the algebra of a supersymmetry quantum system, there are super charge operators Q_i which commute with the Hamiltonian H_{ss} [10]

$$[Q_i, H_{ss}] = 0 \text{ with, } i = 1, 2, 3, \dots, N \quad (1a)$$

and they obey to algebra $\{Q_i, Q_j\} = \delta_{ij} H_{ss}$ (1b)

with H_{ss} is Supersymmetric Hamiltonian. Witten stated that the simplest quantum mechanical system has $N=2$, it was later shown that the case where $N = 1$, if it is supersymmetric, it is equivalent to an $N = 2$ supersymmetric quantum system [7]. In the case where $N = 2$ it is defined that,

$$Q_1 = \frac{1}{\sqrt{2}} \left(\sigma_1 \frac{p}{\sqrt{2m}} + \sigma_2 \phi(x) \right) \text{ and } Q_2 = \frac{1}{\sqrt{2}} \left(\sigma_2 \frac{p}{\sqrt{2m}} + \sigma_1 \phi(x) \right) \quad (2)$$

here the σ_i are the usual Pauli spin matrices, and $p = -i\hbar \frac{\partial}{\partial x}$ is the usual momentum operator. Two

components of SUSY Hamiltonian, we shall write H_{ss} as H_{\pm} , are obtained using equation (1b) and (2) given as,

$$H_{ss} = \begin{pmatrix} -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\hbar}{\sqrt{2m}} \frac{d\phi(x)}{dx} + \phi^2(x) & 0 \\ 0 & -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{\hbar}{\sqrt{2m}} \frac{d\phi(x)}{dx} + \phi^2(x) \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \quad (3)$$

with,

$$H_- = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_-(x) \text{ with } V_-(x) = \phi^2(x) - \frac{\hbar}{\sqrt{2m}} \phi'(x) \quad (4a)$$

and,

$$H_+ = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_+(x) \text{ with } V_+(x) = \phi^2(x) + \frac{\hbar}{\sqrt{2m}} \phi'(x) \quad (4b)$$

with H_- and H_+ are defined as supersymmetry partner in the Hamiltonian, $V_-(x)$ and $V_+(x)$ are the supersymmetry partner each other.

Equations (4a) and (4b) are solved by factorizing the Hamiltonian as

$$H_-(x) = A^+ A, \text{ and } H_+(x) = A A^+ \quad (5)$$

$$\text{where, } A^+ = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + \phi(x) \text{ and } A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + \phi(x) \quad (6)$$

with, A^+ as raising operator, and A as lowering operator.

2.2. Shape Invariance

Gendenshtein [1] discovered another symmetry which if the supersymmetric system satisfies it will be an exactly solvable system, this symmetry is known as shape invariance. If our potential satisfies shape invariance properties we can readily write down its bound state spectrum, and with the help of the charge operators we can find the bound state wave functions [10,14]. It turned out that all the potentials which were known to be exactly solvable until then have the shape invariance symmetry. If the supersymmetric partner potentials have the same dependence on x but differ in a parameter, in such a way that they are related to each other by a change of that parameter, then they are said to be shape invariant. Gendenshtein stated this condition in this way,

$$V_+(x; a_j) = V_-(x; a_{j+1}) + R(a_{j+1}) \quad (7)$$

$$\text{with, } V_+(x; a_j) = \phi^2(x; a_j) + \frac{\hbar}{\sqrt{2m}} \phi'(x; a_j) \quad (8a)$$

$$V_-(x; a_j) = \phi^2(x; a_j) - \frac{\hbar}{\sqrt{2m}} \phi'(x; a_j) \quad (8b)$$

where $j = 0, 1, 2, \dots$, and a is a parameter in our original potential whose ground state energy is zero. $a_{j+1} = f(a_j)$ where f is assumed to be an arbitrary function for the time being. The remainder $R(a_j)$ can be dependent on the parametrization variable a but never on x . In this case V_- is said to be shape invariant, and we can readily find its spectrum, let us take a look at H ,

$$H = H_- + E_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_-(x; a_0) + E_0 \quad (9)$$

According to equations (8b) and (9) we get,

$$V(x) = V_-(x; a_0) + E_0 = \phi^2(x; a_0) - \frac{\hbar}{\sqrt{2m}} \phi'(x; a_0) + E_0 \quad (10)$$

where $V(x)$ is the effective potential V_{eff} , while $\phi(x)$ is determined hypothetically from equation (10) based on the shape of effective potential from the associated system.

By setting $H_- = H_0$ and $H_+ = H_1$ and by applying equations (7), (8a) and (8b) we get the generalized k^{th} hamiltonian equation as,

$$H_k = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_-(x; a_k) + \sum_{i=1}^k R(a_i), \text{ with } k = 0, 1, 2, \dots \quad (11)$$

By applying the characteristic of the Hamiltonian operated to the lowest wave function it is found that $E_0^{(-)} = \sum_{i=1}^k R(a_i)$. So that, in eigen energy spectra of H_- is obtained as,

$$E_n^{(-)} = \sum_{k=1}^n R(a_k) \quad (12)$$

Furthermore, we get the total energy spectra from equation (9) as ,

$$E_n = E_n^{(-)} + E_0 \quad (13)$$

with E_0 as ground state energy of the system.

Based on the characteristics of lowering operator, then the ground state wave function is obtained from equation given as,

$$A \psi_0^{(-)} = 0 \quad (14)$$

Meanwhile, the first excited wave function and so forth $\psi_n^-(x; a_0)$ are obtained by operating raising operator to the ground state wave function $\psi_0^-(x; a_0)$. In general, n^{th} excited state wave function is obtained from the nearest lower wave function given as,

$$\psi_n^{(-)}(x; a_0) \approx A^+(x; a_0) \psi_{n-1}^{(-)}(x; a_1) \quad (15)$$

The explanation above is the simple algebra procedure to construct the hierarchy of Hamiltonian. In the next session, the solution of Schrödinger equation will be investigated by using supercharge operator for either one dimension system or three dimension system which is reduced to one dimension system.

III. Solution of Schrödinger Equation for trigonometric Poschl-Teller potential plus Rosen Morse Non-central potential using Supersymmetry method

Schrödinger equation for trigonometric Poschl-Teller plus Rosen-Morse Non-central potential is the potentials present simultaneously in the quantum system. This non-central potential is expressed as,

$$V(r, \theta) = \frac{\hbar^2}{2m\alpha^2} \left(\frac{\nu(\nu+1)}{\sin^2(r/\alpha)} - 2\mu \cot(r/\alpha) \right) + \frac{\hbar^2}{2mr^2} \left(\frac{a(a-1)}{\sin^2 \theta} + \frac{b(b-1)}{\cos^2 \theta} \right) \quad (16)$$

The three dimensional Schrödinger equation for trigonometric Poschl-Teller plus Rosen-Morse non-central potential is written as,

$$\begin{aligned} & -\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2} \right] \\ & + \frac{\hbar^2}{2m\alpha^2} \left(\frac{\nu(\nu+1)}{\sin^2(r/\alpha)} - 2\mu \cot\left(\frac{r}{\alpha}\right) \right) \psi + \frac{\hbar^2}{2mr^2} \left(\frac{a(a-1)}{\sin^2 \theta} + \frac{b(b-1)}{\cos^2 \theta} \right) \psi = E \psi \end{aligned} \quad (17)$$

If equation (17) multiplied by factor $\left(-\frac{2mr^2}{\hbar^2} \right)$, and then the result is solved using separation variable method

since the non-central potential is separable. By setting $\psi(r, \theta, \varphi) = R(r)P(\theta)\Phi(\varphi)$, with

$RP\Phi(\psi(r, \theta, \varphi) = \frac{X(r)H(\theta)e^{im\varphi}}{r\sqrt{\sin \theta}}$, we obtain,

$$\begin{aligned} & \left[\frac{1}{R} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right) - \frac{r^2}{\alpha^2} \left(\frac{\nu(\nu+1)}{\sin^2\left(\frac{r}{\alpha}\right)} - 2\mu \cot\left(\frac{r}{\alpha}\right) \right) + \frac{2mr^2}{\hbar^2} E \right] \\ & + \left[\frac{1}{P \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial P}{\partial \theta} \right) + \frac{1}{\Phi \sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2} - \left(\frac{a(a-1)}{\sin^2 \theta} + \frac{b(b-1)}{\cos^2 \theta} \right) \right] = 0 \end{aligned} \quad (18)$$

from equation (18) we obtain radial and angular Schrödinger equation as,

$$\begin{aligned} & \frac{1}{R} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right) - \frac{r^2}{\alpha^2} \left(\frac{\nu(\nu+1)}{\sin^2\left(\frac{r}{\alpha}\right)} - 2\mu \cot\left(\frac{r}{\alpha}\right) \right) + \frac{2mr^2}{\hbar^2} E \\ & = \left(\frac{a(a-1)}{\sin^2\theta} + \frac{b(b-1)}{\cos^2\theta} \right) - \frac{1}{P \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial P}{\partial \theta} \right) - \frac{1}{\Phi \sin^2\theta} \frac{\partial^2 \Phi}{\partial \varphi^2} = \ell(\ell+1) \end{aligned} \quad (19)$$

with $l(l+1)$ is constant variabel separable, where ℓ as orbital momentum number.

From equation (19) we get radial and angular Schrödinger equation with single variable as following,

$$\frac{1}{R} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R}{\partial r} \right) - \frac{r^2}{\alpha^2} \left(\frac{\nu(\nu+1)}{\sin^2\left(\frac{r}{\alpha}\right)} - 2\mu \cot\left(\frac{r}{\alpha}\right) \right) + \frac{2mr^2}{\hbar^2} E = \ell(\ell+1) \quad (20)$$

or equation (20) multiplied by $\left(\frac{R}{r^2}\right)$, with $R(r) = \frac{\chi(r)}{r}$, so using symple algebra, we get,

$$\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{\alpha^2} \left(\frac{\nu(\nu+1)}{\sin^2\left(\frac{r}{\alpha}\right)} - 2\mu \cot\left(\frac{r}{\alpha}\right) \right) \chi - \frac{\ell(\ell+1)}{r^2} \chi = -\frac{2m}{\hbar^2} E \chi \quad (21)$$

and than, for solve radial Schrödinger equation, we use approximation for centrifugal term,

$$\frac{1}{r^2} \cong \frac{1}{\alpha^2} \left(d_0 + \frac{1}{\sin^2\left(\frac{r}{\alpha}\right)} \right) \text{ for } \frac{r}{\alpha} \ll 1, \text{ with } d_0 = \frac{1}{12}, \epsilon^2 = -\frac{2m}{\hbar^2} E, \text{ we get,}$$

$$\frac{\partial^2 \chi}{\partial r^2} - \frac{1}{\alpha^2} \left(\frac{\nu(\nu+1)}{\sin^2\left(\frac{r}{\alpha}\right)} - 2\mu \cot\left(\frac{r}{\alpha}\right) \right) \chi - \ell(\ell+1) \frac{1}{\alpha^2} \left(d_0 + \frac{1}{\sin^2\left(\frac{r}{\alpha}\right)} \right) \chi = \epsilon^2 \chi \quad (22)$$

from equation (22) simplied by $\left(-\frac{\hbar^2}{2m}\right)$ we get radial Schrödinger equation,

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \chi}{\partial r^2} + \frac{\hbar^2}{2m\alpha^2} \left(\frac{\nu(\nu+1) + \ell(\ell+1)}{\sin^2\left(\frac{r}{\alpha}\right)} - 2\mu \cot\left(\frac{r}{\alpha}\right) \right) \chi + \frac{\hbar^2}{2m\alpha^2} \ell(\ell+1) d_0 \chi = -\frac{\hbar^2}{2m} \epsilon^2 \chi \quad (23)$$

The angular Schrödinger equation obtained from equation (19) is given as,

$$\left(\frac{a(a-1)}{\sin^2\theta} + \frac{b(b-1)}{\cos^2\theta} \right) - \frac{1}{P \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial P}{\partial \theta} \right) - \frac{1}{\Phi \sin^2\theta} \frac{\partial^2 \Phi}{\partial \varphi^2} = \ell(\ell+1) \quad (24)$$

and by setting $\frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \varphi^2} = -m^2$ we get azimuthal wave function as

$$\Phi = \sqrt{\frac{1}{2\pi}} e^{im\varphi}, m = 0, \pm 1, \pm 2, \dots \quad (25)$$

and equation (24) becomes,

$$\left(\frac{a(a-1)}{\sin^2\theta} + \frac{b(b-1)}{\cos^2\theta} \right) - \frac{1}{P \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial P}{\partial \theta} \right) + \frac{m^2}{\sin^2\theta} = \ell(\ell+1) \quad (26)$$

with m^2 as variable separation and we get one dimensional angular Schrödinger equation,

$$-\frac{\hbar^2}{2m} \frac{d^2 H}{d\theta^2} + \frac{\hbar^2}{2m} \left(\frac{a(a-1) + m^2 - \frac{1}{4}}{\sin^2\theta} + \frac{b(b-1)}{\cos^2\theta} \right) H = \frac{\hbar^2}{2m} (\ell(\ell+1) + \frac{1}{4}) H \quad (27)$$

3.1. The solution of Radial Scrodinger Equation for trigonometric Poschl teller potential plus Rosen Morse

Factor R in equation (20) is defined as wave function ψ , then the Schrödinger equation for trigonometric Poschl-Teller plus Rosen-Morse potential in radial with the assumption of $\epsilon' = -\frac{\hbar^2}{2m} \epsilon^2$ can be rewritten as follow,

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \chi}{\partial r^2} + \frac{\hbar^2}{2m\alpha^2} \left(\frac{\nu(\nu+1) + \ell(\ell+1)}{\sin^2\left(\frac{r}{\alpha}\right)} - 2\mu \cot\left(\frac{r}{\alpha}\right) \right) \chi + \frac{\hbar^2}{2m\alpha^2} \ell(\ell+1) d_0 \chi = \epsilon' \chi \quad (28)$$

Based on equation (28), the effective potential of radial SE trigonometric Poschl-Teller plus Rosen-Morse is rewritten as,

$$V_{eff} = \frac{\hbar^2}{2m\alpha^2} \left(\frac{\nu(\nu+1) + \ell(\ell+1)}{\sin^2\left(\frac{r}{\alpha}\right)} - 2\mu \cot\left(\frac{r}{\alpha}\right) \right) + \frac{\hbar^2}{2m\alpha^2} \ell(\ell+1) d_0 \quad (29)$$

or,

$$V_{eff} = \frac{\hbar^2}{2m\alpha^2} \left(\frac{\nu'(\nu'+1)}{\sin^2\left(\frac{r}{\alpha}\right)} - 2\mu \cot\left(\frac{r}{\alpha}\right) \right) + \frac{\hbar^2}{2m\alpha^2} \ell(\ell+1) d_0 \quad (30)$$

$$\text{with } \nu' = \sqrt{\nu(\nu+1) + \ell(\ell+1) + \frac{1}{4}} - \frac{1}{2}$$

By inserting effective potential in equation (30) into equation (10), we obtain

$$\phi^2(x) - \frac{\hbar}{\sqrt{2m}} \phi'(x) = \frac{\hbar^2}{2m\alpha^2} \left(\frac{\nu'(\nu'+1)}{\sin^2(\frac{r}{\alpha})} - 2\mu \cot(\frac{r}{\alpha}) \right) + \frac{\hbar^2}{2m\alpha^2} \ell(\ell+1)d_0 - \varepsilon \quad (31)$$

with ε is factorization energy or ground state energy of the system. From equation (31) it is intellectually guessed that superpotential in equation (30) is proposed as,

$$\phi(x) = A \cot(\frac{r}{\alpha}) - \frac{B}{A} \quad (32)$$

where A and B are indefinite constants that will be calculated. From equation (32), we can determine the value of $\phi'(x)$ and $\phi^2(x)$, then the result is inserted into equation (31) and we get

$$\frac{A^2}{\sin^2(\frac{r}{\alpha})} - A^2 - 2B \cot(\frac{r}{\alpha}) + \frac{B^2}{A^2} + \frac{\hbar}{\sqrt{2m}} \frac{A}{\alpha} \frac{1}{\sin^2(\frac{r}{\alpha})} = \frac{\hbar^2}{2m\alpha^2} \left(\frac{\nu'(\nu'+1)}{\sin^2(\frac{r}{\alpha})} - 2\mu \cot(\frac{r}{\alpha}) \right) + \frac{\hbar^2}{2m\alpha^2} \ell(\ell+1)d_0 - \varepsilon \quad (33)$$

By analysing the similar term between left and right hand side in equation (33), we obtain,

$$\left(A^2 + \frac{\hbar}{\alpha\sqrt{2m}} A \right) \frac{1}{\sin^2(\frac{r}{\alpha})} = \frac{\hbar^2}{2m\alpha^2} (\nu'(\nu'+1)) \frac{1}{\sin^2(\frac{r}{\alpha})}; -B \cot(\frac{r}{\alpha}) = -\frac{\hbar^2}{2m\alpha^2} \mu \cot(\frac{r}{\alpha});$$

$$\text{and } -A^2 + \frac{B^2}{A^2} = \frac{\hbar^2}{2m\alpha^2} \ell(\ell+1)d_0 - \varepsilon \quad (34)$$

From the three equation in equation (34), it is obtained that,

$$A_+ = \frac{\hbar}{\alpha\sqrt{2m}} \nu' \text{ and } A_- = -\frac{\hbar}{\alpha\sqrt{2m}} (\nu'+1) \quad (35a)$$

$$B = \frac{\hbar^2}{2m\alpha^2} \mu \quad (35b)$$

$$E_0 = \varepsilon = -\frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+1)^2} - (\nu'+1)^2 - \ell(\ell+1)d_0 \right) \quad (35c)$$

The value of A and B are chosen by considering that the value of $E_0^{(-)}$ is equal to zero, so,

$$\phi(r) = -\frac{\hbar}{\alpha\sqrt{2m}} \left\{ (\nu'+1) \cot(\frac{r}{\alpha}) - \frac{\mu}{(\nu'+1)} \right\} \quad (36)$$

By using equations (6) and (36), we get

$$A^+ = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dr} + \phi(r) = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dr} - \frac{\hbar}{\alpha\sqrt{2m}} \left((\nu'+1) \cot(\frac{r}{\alpha}) - \frac{\mu}{(\nu'+1)} \right) \quad (37)$$

$$\text{and } A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dr} + \phi(r) = \frac{\hbar}{\sqrt{2m}} \frac{d}{dr} - \frac{\hbar}{\alpha\sqrt{2m}} \left((\nu'+1) \cot(\frac{r}{\alpha}) - \frac{\mu}{(\nu'+1)} \right) \quad (38)$$

The ground state wave function is obtained from equation (14) and (38) given as,

$$\left\{ \frac{\hbar}{\sqrt{2m}} \frac{d}{dr} + \phi(r) \right\} \psi_0^- = 0$$

that gives,

$$\int \frac{d\psi_0^-}{\psi_0^-}(r, a_0) = (\nu'+1) \int \cot(\frac{r}{\alpha}) d(\frac{r}{\alpha}) - \frac{\mu}{\alpha(\nu'+1)} \int dr$$

$$\ln \psi_0^-(r, a_0) = \left[(\nu'+1) \ln \sin(\frac{r}{\alpha}) - \frac{\mu}{\alpha(\nu'+1)} r \right] + C$$

$$\psi_0^-(r, a_0) = C \left(\sin(\frac{r}{\alpha}) \right)^{(\nu'+1)} \exp \left(-\frac{\mu}{\alpha(\nu'+1)} r \right) \quad (39)$$

By using equation (15) we obtain the first excited wave function as,

$$\psi_1^{(-)}(r; a_0) \approx A^+(x; a_0) \psi_0^{(-)}(r; a_1) \quad (40)$$

where $a_0 = \nu' = a_1 = \nu'+1, \dots, a_n = \nu'+n$ is the independent parameter to variable "r". By inserting the value of the parameter to equation (39) and (37) and by using equation (40), we get

$$\begin{aligned} \psi_1^{(-)}(r; a_0) &\approx \left(-\frac{\hbar}{\sqrt{2m}} \frac{d}{dr} - \frac{\hbar}{\alpha\sqrt{2m}} \left((\nu'+1) \cot\left(\frac{r}{\alpha}\right) - \frac{\mu}{(\nu'+1)} \right) \right) \left\{ C \left(\sin\left(\frac{r}{\alpha}\right) \right)^{(\nu'+2)} \exp\left(-\frac{\mu}{\alpha(\nu'+2)} r\right) \right\} \\ &\approx \frac{\hbar}{\alpha\sqrt{2m}} \left(\left(\frac{(2\nu'+3)\mu}{(\nu'+2)(\nu'+1)} \sin\left(\frac{r}{\alpha}\right) - (2\nu'+3) \cos\left(\frac{r}{\alpha}\right) \right) \right) \left\{ C \left(\sin\left(\frac{r}{\alpha}\right) \right)^{(\nu'+1)} \exp\left(-\frac{\mu}{\alpha(\nu'+2)} r\right) \right\} \end{aligned} \quad (41)$$

By repeating the step in obtaining equation (41) we get the upper levels of excited wave function as

$\psi_2^{(-)}(r; a_0), \psi_3^{(-)}(r; a_0), \dots$ and so on.

The ground state radial wave functions with various value of the orbital quantum number n_ℓ but with the certain values of potential parameters are shown in Table 1.

Table 1. ground state radial wave function for trigonometric Poschl-Teller plus Rosen-Morse non-central potential with different orbital quantum number

n_r	a	b	m	μ	n_ℓ	ν	ℓ	ν'	$\psi_{(n_r; \ell; \nu')}^{(-)}$
0	4	2	1	1	0	2	5,6	6,08	$(\sin(r))^{7.08} e^{\left(-\frac{r}{7.08}\right)}$
0	4	2	1	1	1	2	7,6	7,97	$(\sin(r))^{8.97} e^{\left(-\frac{r}{8.97}\right)}$
0	4	2	1	1	2	2	9,6	9,9	$(\sin(r))^{10.9} e^{\left(-\frac{r}{10.9}\right)}$

The radial wave function of trigonometric Poschl Teller plus Rosen Morse non central potential, it seems only affected by r function that described distant motion or imminent electron from atom core. Polar quantum number n_ℓ , μ and magnetic quantum number m constant, at this point this research was focused on the same electron, therefore, the increasing of disturbance from Rosen Morse non central potential from the value of ℓ , and ν' that getting bigger showed the increasing hyperbolic synus factor of wave function, mathematically, it will be affected to the shift of wave function. The shift that occurred can be seen in radial wave function side visualisation on the figure 1. The effect of Poschl-Teller potential to the radial wave function determined by the values of l . The larger is the values of l , the larger is the values of ν' which causes the increase in the wave amplitudes, as shown in Figure 1. The effect is larger for larger r .

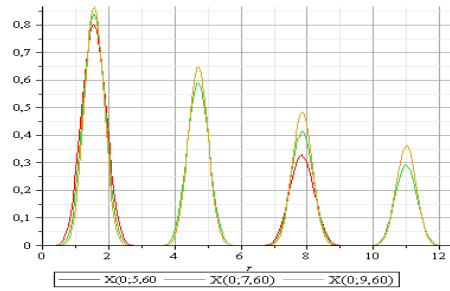


Figure 1. The ground state radial wave function for trigonometric Poschl-Teller plus Rosen Morse non central potential for different n_ℓ

Figure 1. Show that effect of the Poschl Teller non central potential result in the shifting of wave function toward the radial direction r , thus increasing wave amplitudo. The shifting that occurred can be seen on the visualisation of wave function radial side on figure 1, it can be seen that the effect of Poschl Teller non central potential results wave function shift towards radial direction r , but not too significant, then it increasing amplitudo wave. The shift of amplitudo indicates the energy level that getting bigger then electron by way motion that not too far, which indicates the probability of finding bigger electron. By using equations (8a), (8b) and (36), we determine the potential partner which have shape invariant as

$$V_-(r, a_0) = \frac{\hbar^2}{2m\alpha^2} \frac{\nu'(\nu'+1)}{\sin^2\left(\frac{r}{\alpha}\right)} - \frac{\hbar^2}{m\alpha^2} \mu \cot\left(\frac{r}{\alpha}\right) + \frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+1)^2} - (\nu'+1)^2 \right) \quad (42a)$$

and,

$$V_+(r, a_0) = \frac{\hbar^2}{2m\alpha^2} \frac{(\nu'+1)(\nu'+2)}{\sin^2(\frac{r}{\alpha})} - \frac{\hbar^2}{m\alpha^2} \mu \cot(\frac{r}{\alpha}) + \frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+1)^2} - (\nu'+1)^2 \right) \quad (42b)$$

If we have chosen parameters $a_0 = \nu', a_1 = \nu' + 1, a_2 = \nu' + 2, \dots$ then $V_-(r, a_1)$

obtained by changing ν' into $\nu'+1$ in equation (42a) is given as

$$V_-(r, a_1) = \frac{\hbar^2}{2m\alpha^2} \frac{(\nu'+1)(\nu'+2)}{\sin^2(\frac{r}{\alpha})} - \frac{\hbar^2}{m\alpha^2} \mu \cot(\frac{r}{\alpha}) + \frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+2)^2} - (\nu'+2)^2 \right) \quad (43)$$

From those two equations (42b) and (43) can be seen that $V_+(r, a_0)$ have similar shape with $V_-(r, a_1)$, and by using shape invariance relation in equation (8) we obtain $R(a_1)$ as

$$R(a_1) = V_+(r, a_0) - V_-(r, a_1) = \frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+1)^2} - (\nu'+1)^2 \right) - \frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+2)^2} - (\nu'+2)^2 \right) \quad (44)$$

We repeat the step as on the determination of equation (44) by using equations (42a), (42b), and (43), to obtain $V_+(r, a_1)$ and $V_-(r, a_2)$, as,

$$V_+(r, a_1) = \frac{\hbar^2}{2m\alpha^2} \frac{(\nu'+2)(\nu'+3)}{\sin^2(\frac{r}{\alpha})} - \frac{\hbar^2}{m\alpha^2} \mu \cot(\frac{r}{\alpha}) + \frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+2)^2} - (\nu'+2)^2 \right) \quad (45a)$$

$$V_-(r, a_2) = \frac{\hbar^2}{2m\alpha^2} \frac{(\nu'+2)(\nu'+3)}{\sin^2(\frac{r}{\alpha})} - \frac{\hbar^2}{m\alpha^2} \mu \cot(\frac{r}{\alpha}) + \frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+3)^2} - (\nu'+3)^2 \right) \quad (45b)$$

From equ. (45a) and (45b) so obtained,

$$R(a_2) = V_+(r, a_1) - V_-(r, a_2) = \frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+2)^2} - (\nu'+2)^2 \right) - \frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+3)^2} - (\nu'+3)^2 \right) \quad (46)$$

Then, the determination steps on equ. (44) or equ. (46) above are repeated until parameters heading to n, a_n to determinate $R(a_n)$ and finally obtained,

$$E_n^{(-)} = \sum_{k=1}^n R(a_k) = \frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+1)^2} - (\nu'+1)^2 \right) - \frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+n_r+1)^2} - (\nu'+n_r+1)^2 \right) \quad (47)$$

If equ. (47) and equ. (36c) inserted into equation (13) we obtain the energy spectrum for Rosen Morse system i.e.,

$$E_n = E_n^{(-)} + E_0 = -\frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\nu'+n_r+1)^2} - (\nu'+n_r+1)^2 - \ell(\ell+1)d_0 \right) \quad (48)$$

with, $E_n = \epsilon^n$, and $\nu' = \sqrt{\nu'(\nu'+1) + \ell(\ell+1) + \frac{1}{4}} - \frac{1}{2}$, so equ. (48) is energy solution of Schrödinger equation for trigonometric Poschl-Teller plus Rosen Morse non central potential with

$$-\frac{\hbar^2}{2m} \epsilon^n = E_n$$

$$-\frac{\hbar^2}{2m} \left(-\frac{2m}{\hbar^2} E \right) = -\frac{\hbar^2}{2m\alpha^2} \left(\frac{\mu^2}{(\sqrt{\nu'(\nu'+1) + \ell(\ell+1) + \frac{1}{4}} - \frac{1}{2} + n_r + 1)^2} - (\sqrt{\nu'(\nu'+1) + \ell(\ell+1) + \frac{1}{4}} - \frac{1}{2} + n_r + 1)^2 - \ell(\ell+1)d_0 \right)$$

$$E_{nr} = \frac{\hbar^2}{2m\alpha^2} \left(\left(\sqrt{\nu'(\nu'+1) + \ell(\ell+1) + \frac{1}{4}} + n_r + \frac{1}{2} \right)^2 + \ell(\ell+1)d_0 - \frac{\mu^2}{\left(\sqrt{\nu'(\nu'+1) + \ell(\ell+1) + \frac{1}{4}} + n_r + \frac{1}{2} \right)^2} \right) \quad (49)$$

Equation (49) showed energy spectra of trigonometric Poschl-Teller plus Rosen Morse non central potential, with,

\hbar : planck constants,

m : elementary particle mass

a and b : constants potential depth,

n : principle quantum numbers, $n=1,2,3,\dots$

n_r : radial quantum numbers, $n_r=0,1,2,\dots$

l : orbital quantum numbers.

From equation (49), we draw the graph of the energy spectra of trigonometric Poschl Teller plus Rosen Morse non central potential for different level shown in Figure 2.

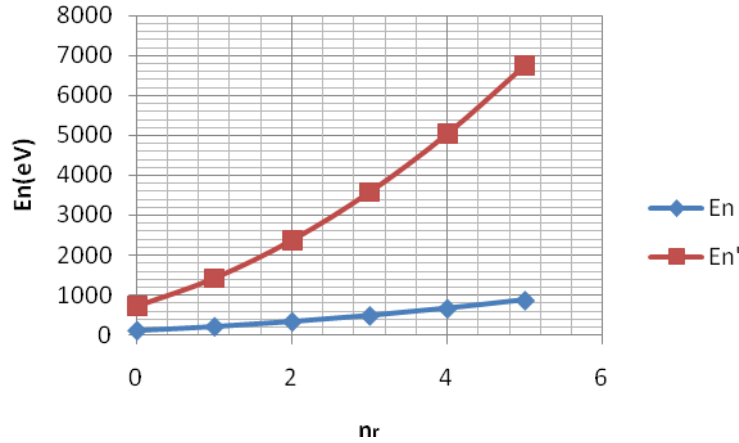


Figure 2. The Energy spectra graph of Rosen Morse potential, with and without the presence of Poschl-Teller potential, with $m=1$, $n_l = 1$, $\mu=1$, $\nu=2$, and $n_r = 0,1,2,3,4,5$.

Figure 2. shows that the value of energies affected by quantum orbital number l which depends on the values of Pochl-Teller potential's parameter.

3.2 The solution of angular Schrödinger equation trigonometric Poschl-Teller plus Rosen Morse non-central potential.

To ease the solution of angular Schrödinger Equation, i.e.,

$$\frac{\hbar^2}{2m}(\ell(\ell+1) + \frac{1}{4})H = EH \quad (50)$$

If equ. (50) incorporated to equ. (28) so angular Schrödinger equation Poschl-Teller plus Rosen Morse non central potential changed into,

$$-\frac{\hbar^2}{2m} \frac{d^2 H}{d\theta^2} + \frac{\hbar^2}{2m} \left(\frac{a(a-1) + m^2 - \frac{1}{4}}{\sin^2 \theta} + \frac{b(b-1)}{\cos^2 \theta} \right) H = EH \quad (51)$$

Based on equ. (51), effective potential of angular trigonometric Poschl-Teller plus Rosen-Morse non central potential is expressed as,

$$V_{eff} = \frac{\hbar^2}{2m} \left(\frac{a(a-1) + m^2 - \frac{1}{4}}{\sin^2 \theta} + \frac{b(b-1)}{\cos^2 \theta} \right) \quad (52)$$

if, $a(a-1) + m^2 - \frac{1}{4} = a'(a'-1)$, we get

$$V_{eff} = \frac{\hbar^2}{2m} \left(\frac{a'(a'-1)}{\sin^2 \theta} + \frac{b(b-1)}{\cos^2 \theta} \right) \quad (53)$$

with $a' = \sqrt{a(a-1) + m^2} + \frac{1}{2}$

According to the form of those effective potential equations, then superpotential equation of angular trigonometric Poschl-Teller plus Rosen-Morse non central potential is expressed as,

$$\phi(\theta) = A \tan \theta + B \cot \theta \quad (54)$$

where A and B are indefinite constant that will be calculated. From equ. (54), we determine the value of $\phi_0'(x)$ and $\phi_0^2(x)$, and by using equation (6) we have

$$A^2 \tan^2 \theta + B^2 \cot^2 \theta + 2AB - \frac{\hbar}{\sqrt{2m}} (A \sec^2 \theta - B \csc^2 \theta) = \frac{\hbar^2}{2m} \left(\frac{a'(a'-1)}{\sin^2 \theta} + \frac{b(b-1)}{\cos^2 \theta} \right) - \varepsilon \quad (55)$$

By using in common concept of coefficient between left and right hand side, we obtain

$$\left(A^2 + \frac{\hbar}{\sqrt{2m}} A \right) \frac{1}{\cos^2 \theta} = \frac{\hbar^2}{2m} (b(b-1)) \frac{1}{\cos^2 \theta}; \left(B^2 + \frac{\hbar}{\sqrt{2m}} B \right) \frac{1}{\sin^2 \theta} = \frac{\hbar^2}{2m\alpha^2} a'(a'-1) \frac{1}{\sin^2 \theta};$$

$$\text{and } \varepsilon = E_0 = A^2 + B^2 - 2AB \quad (56)$$

from those third equations on equ. (56) is obtained,

$$A_+ = \frac{\hbar}{\sqrt{2m}} b \quad \text{atau} \quad A_- = -\frac{\hbar}{\sqrt{2m}} (b-1)$$

(57a)

$$B_+ = \frac{\hbar}{\sqrt{2m}}(a'-1) = \frac{\hbar}{\sqrt{2m}}\sqrt{a(a-1)+m^2} - \frac{1}{2}$$

$$\text{or } B_- = -\frac{\hbar}{\sqrt{2m}}a' = -\frac{\hbar}{\sqrt{2m}}\sqrt{a(a-1)+m^2} + \frac{1}{2} \quad (57b)$$

$$E_0 = \varepsilon = \frac{\hbar^2}{2m}(b+a')^2 \quad (57c)$$

A and B value are chosen so that the value of $E_0^{(-)}$ is zero.

By using eqs. (8a) and (8b) are obtained,

$$\phi(\theta) = \frac{\hbar}{\sqrt{2m}}b \tan \theta - \frac{\hbar}{\sqrt{2m}}a' \cot \theta \quad (58)$$

$$\begin{aligned} V_-(\theta; a'_0 b_0) &= \phi^2(\theta; a'_0 b_0) - \frac{\hbar}{\sqrt{2m}}\phi'(\theta; a'_0 b_0) \\ &= \frac{\hbar^2}{2m}\left(\frac{a'(a'-1)}{\sin^2 \theta} + \frac{b(b-1)}{\cos^2 \theta}\right) - \frac{\hbar^2}{2m}(b+a')^2 \end{aligned} \quad (59a)$$

$$\begin{aligned} V_+(\theta; a'_0 b_0) &= \phi^2(\theta; a'_0 b_0) + \frac{\hbar}{\sqrt{2m}}\phi'(\theta; a'_0 b_0) \\ &= \frac{\hbar^2}{2m}\left(\frac{a'(a'+1)}{\sin^2 \theta} + \frac{b(b+1)}{\cos^2 \theta}\right) - \frac{\hbar^2}{2m}(b+a')^2 \end{aligned} \quad (59b)$$

From those two equations (59a) and (59b) is obtained $a_0 = a'; b_0 = b; a_1 = a' + 1; b_1 = b + 1; \dots$

$$V_-(\theta; a'_1 b_1) = \frac{\hbar^2}{2m}\left(\frac{a'(a'+1)}{\sin^2 \theta} + \frac{b(b+1)}{\cos^2 \theta}\right) - \frac{\hbar^2}{2m}(b+a'+2)^2 \quad (60)$$

From those two equ. (59b) and (60) can be seen that $V_+(\theta, a_0 b_0)$ have the same form with $V_-(\theta, a_1 b_1)$, and by using *shape invariance* relation on equ. (8), is obtained that $R(a_1 b_1)$

$$R(a'_1 b_1) = V_+(\theta; a'_0 b_0) - V_-(\theta; a'_1 b_1) = \frac{\hbar^2}{2m}(b+a'+2)^2 - \frac{\hbar^2}{2m}(b+a')^2 \quad (61)$$

We repeated the step as on determination of equ. (61) with using steps equ. (59), and equ. (60) to obtain $V_+(\theta, a_1 b_1)$ and $V_-(\theta, a_2 b_2)$ equations, so obtained,

$$V_+(\theta; a'_1 b_1) = \frac{\hbar^2}{2m}\left(\frac{(a'+1)(a'+2)}{\sin^2 \theta} + \frac{(b+1)(b+2)}{\cos^2 \theta}\right) - \frac{\hbar^2}{2m}(b+a'+2)^2 \quad (62a)$$

$$V_-(\theta; a'_2 b_2) = \frac{\hbar^2}{2m}\left(\frac{(a'+1)(a'+2)}{\sin^2 \theta} + \frac{(b+1)(b+2)}{\cos^2 \theta}\right) - \frac{\hbar^2}{2m}(b+a'+4)^2 \quad (62b)$$

By repeated the step from equ. (62a) to (62b) we often,

$$R(a'_2 b_2) = V_+(\theta; a'_1 b_1) - V_-(\theta; a'_2 b_2) = \frac{\hbar^2}{2m}(b+a'+4)^2 - \frac{\hbar^2}{2m}(b+a'+2)^2 \quad (63)$$

$$R(a'_n b_n) = V_+(\theta; a'_{n-1} b_{n-1}) - V_-(\theta; a'_n b_n) = \frac{\hbar^2}{2m}(b+a'+2n)^2 - \frac{\hbar^2}{2m}(b+a'+2n-2)^2 \quad (64)$$

By repeating the steps in equ. (61) or equ. (63) until parameters heading to n, $a_n b_n$ to obtain $R(a_n b_n)$ as on equ. (64) and finally it is obtained the parameter that has energy order given as,

$$E_n^{(-)} = \sum_{k=1}^n R(a_k) = \frac{\hbar^2}{2m}(b+a'+2n)^2 - \frac{\hbar^2}{2m}(b+a')^2 \quad (65)$$

If equ. (65) and equ. (57c) are inserted into equation (13) we obtain energy spectrum of Poschl-Teller system so,

$$E_n = E_n^{(-)} + E_0 = \frac{\hbar^2}{2m}(b+a'+2n)^2 - \frac{\hbar^2}{2m}(b+a')^2 + \frac{\hbar^2}{2m}(b+a')^2$$

$$\text{so } E_n = \frac{\hbar^2}{2m}(b+a'+2n)^2 \quad (66)$$

$$\text{with } a' = \sqrt{a(a-1)+m^2} + \frac{1}{2}$$

By using the parameter that has the order the same with energy expressed in equation (50) we obtain the orbital quantum number given as

$$(\ell(\ell+1) + \frac{1}{4}) = \left(\sqrt{a(a-1) + m^2} + \frac{1}{2} + b + 2n \right)^2$$

$$\text{or } \ell = \sqrt{a(a-1) + m^2} + b + 2n \quad (67)$$

The angular quantum numbers on equation (67) is used to calculate energy spectrum for potential non central system.

By using equ. (6) and (58) are obtained

$$A^+ = -\frac{\hbar}{\sqrt{2m}} \frac{d}{d\theta} + \phi(\theta) = -\frac{\hbar}{\sqrt{2m}} \frac{d}{d\theta} + \frac{\hbar}{\sqrt{2m}} b \tan \theta - \frac{\hbar}{\sqrt{2m}} a' \cot \theta \quad (68a)$$

$$\text{and } A = \frac{\hbar}{\sqrt{2m}} \frac{d}{d\theta} + \phi(\theta) = \frac{\hbar}{\sqrt{2m}} \frac{d}{d\theta} + \frac{\hbar}{\sqrt{2m}} b \tan \theta - \frac{\hbar}{\sqrt{2m}} a' \cot \theta \quad (68b)$$

By using lowering operator on equ (68b), we calculate the ground state wave function for angular trigonometric Poschl-Teller plus Rosen Morse non-central potential as follows,

$$\left\{ \frac{\hbar}{\sqrt{2m}} \frac{d}{d\theta} + \frac{\hbar}{\sqrt{2m}} b \tan \theta - \frac{\hbar}{\sqrt{2m}} a' \cot \theta \right\} \psi_0^- = 0$$

$$\int \frac{d\psi_0^-}{\psi_0^-} (\theta, a_0) = a' \int \cot \theta d\theta - b \int \tan \theta d\theta$$

$$\psi_0^-(r, a_0) = C((\cos \theta)^b (\sin \theta)^{a'}) = C((\cos \theta)^b (\sin \theta)^{\sqrt{a(a-1)+m^2} + \frac{1}{2}}) \quad (69)$$

Then, by using increasing operator on equ (68a) and basic wave function determined first level excited wave function,

$$\psi_1^{(-)}(\theta; a_0) \approx A^+(\theta; a_0) \psi_0^{(-)}(\theta, a_1)$$

$$\begin{aligned} \psi_1^{(-)}(\theta; a_0) &= \left(-\frac{\hbar}{\sqrt{2m}} \frac{d}{d\theta} + \frac{\hbar}{\sqrt{2m}} b \tan \theta - \frac{\hbar}{\sqrt{2m}} a' \cot \theta \right) ((\cos \theta)^{b+1} (\sin \theta)^{a'+1}) \\ &= \frac{\hbar}{\sqrt{2m}} \left\{ (2b+1)(\sin \theta)^2 - (2a'+1)(\cos \theta)^2 \right\} (\cos \theta)^b (\sin \theta)^{a'} \end{aligned} \quad (70)$$

$$\begin{aligned} \psi_2^{(-)}(\theta; a_0) &= \frac{\hbar^2}{2m} \left(-\frac{d}{d\theta} + b \tan \theta - a' \cot \theta \right) \left\{ (2b+3)(\sin \theta)^2 - (2a'+3)(\cos \theta)^2 \right\} (\cos \theta)^{b+1} (\sin \theta)^{a'+1} \\ &= \frac{\hbar^2}{2m} \left\{ (4a'^2 + 8a'+3)(\cos \theta)^4 + (4b^2 + 8b + 3)(\sin \theta)^4 \right\} - (8ba' + 12a' + 12b)(\sin \theta)^2 (\cos \theta)^2 \\ &\quad \left((\cos \theta)^b (\sin \theta)^{a'} \right) \end{aligned} \quad (71)$$

To determine the excited wave function above can be done as on determination of first level excited wave function as follows,

$$\psi_2^{(-)}(\theta; a_0), \psi_3^{(-)}(\theta; a_0), \text{ and so on.}$$

Therefore obtained wave function level that is wanted.



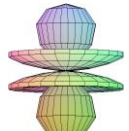

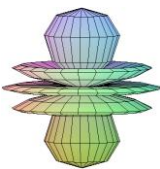
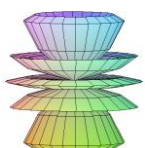
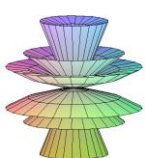
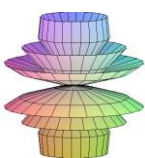
Since $Q_n^{(-)}(x; a_n) = \psi_n^{(-)}(x; a_n)$, so we get,

$$P = \frac{\psi_n^{(-)}(\theta; a_0)}{\sqrt{\sin \theta}} \quad (72)$$

$$\text{with } a' = \sqrt{a(a-1) + m^2} + \frac{1}{2}$$

Furthermore, the result of each calculation from 1st and 2nd excited wave function, with the value n_1 , m, a, and b, which is listed in table 2, at the same time altogether with the visualisation of polar wave function shown on table 2, below,

Table 2. Polar wave function of trigonometric Poschl-Teller potential plus Rosen Morse

No	$P_{n_1, m, a, b}$	n_1	m	a	b	a'	$P = \frac{\psi_n^{(-)}(\theta; a_n)}{\sqrt{\sin \theta}}$
1	P_{1000}	1	0	0	0	0,5	$\{(\sin \theta)^2 - 2(\cos \theta)^2\}$ 
2	P_{1100}	1	1	0	0	1,5	$\{(\sin \theta)^2 - 4(\cos \theta)^2\}(\sin \theta)$ 
3	P_{1002}	1	0	0	2	0,5	$\{5(\sin \theta)^2 - 2(\cos \theta)^2\}(\cos \theta)^2$ 
4	P_{1024}	1	0	2	4	1,91	$\{9.(\sin \theta)^2 - (4,83)(\cos \theta)^2\}(\cos \theta)^4 (\sin \theta)^{1,41}$ 
5	P_{2000}	2	0	0	0	0,5	$8.(\cos \theta)^4 + 3.(\sin \theta)^4 - 24.(\sin \theta)^2 (\cos \theta)^2$ 
6	P_{2120}	2	1	2	0	2,23	$\{(40,78)(\cos \theta)^4 + 3.(\sin \theta)^4 - (44,78).(\sin \theta)^2.(\cos \theta)^2\}(\sin \theta)^{1,7}$ 
7	P_{2104}	2	1	0	4	1,5	$\{(24)(\cos \theta)^4 + 99.(\sin \theta)^4 - (132).(\sin \theta)^2.(\cos \theta)^2\}(\cos \theta)^4 (\sin \theta)$ 
8	P_{2124}	2	1	2	4	2,23	$\{(40,78)(\cos \theta)^4 + 99.(\sin \theta)^4 - (164,2).(\sin \theta)^2.(\cos \theta)^2\}(\cos \theta)^4 (\sin \theta)^{1,73}$ 

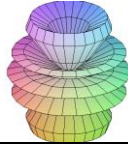

9	P_{2000}	2	0	2	0	1,91	$\{(32,97)(\cos\theta)^4 + 3.(\sin\theta)^4 - (40,9).(\sin\theta)^2.(\cos\theta)^2\}$ $(\sin\theta)^{1,4}$ 
10	P_{2002}	2	0	0	2	0,5	$\{(8)(\cos\theta)^4 + 35.(\sin\theta)^4 - (56).(\sin\theta)^2.(\cos\theta)^2\}$ $(\cos\theta)^2$ 

Table 2. shown the form of polar wave function connected to the angular spin direction electron momentum also describe solid dependent probability on the angular. Generally, polar wave function definition is same with radial wave function which is describe the probability of electron finding, but both of it have difference on its motion; if radial wave function connected to the far or near of the electron to the nucleus, so polar wave function is connected to the rotation of the electron to the nucleus.

IV. Results And Discussion

The radial wave function of trigonometric Poschl Teller plus Rosen Morse non-central potential seems only affected by r function that describe far or near electron motion from the atom core. Orbital quantum numbers n_l , μ , and magnetic quantum numbers constants m , on this research is focused on the same electron. Hence to that, the increasing effect of Rosen Morse non central potential of l , and v' value that getting bigger show the increasing hyperbolic synus factor of wave function, mathematically, will be affected to the shift of wave function. The shift that occurred can be seen on the visualisation of radial ground state wave function on Figure 1, it shown that trigonometric Poschl Teller potential results on the shift of wave function towards r radial direction, but not too significant, at the same time increase wave amplitudo. The shift of amplitudo indicates the level of energy that getting bigger also electron byway motion that is not too distant, which indicates the probability of bigger electron found. Equation (49), is obtained energy spectra graphic of trigonometric Poschl Teller plus Rosen Morse non central potential, can be seen that value of orbital quantum numbers affected by l value, with m , n_l , μ dan v constants. The bigger l value determinate capacity of effect from Poschl Teller potential, at this point, the bigger l value, Rosen Morse potential will experience the bigger effect from Poschl Teller potential, so that the electrons need higher energy to be on certain layer. On table 2, showed that parameter a' and b influencing wave function. Value of b gives exponential factor to the wave function while a' value increasing sinusoidal factor wave function. At this matter, value of polar quantum number factor (n_l) gives influence on the complex fraction of angular θ function. Parameter a' which is affected by factor 'm' value that break down angular θ function with little angular φ function, b parameter breaks down angular θ function with big angular φ function. Figure from table 2, show that there are close and open wave visualisation results. Those results are affected by the existence of sinusoidal factor on the wave function that were used. It can be seen that if synus factor on wave function equation have value, then the resulted wave visualisation is open, and on the contrary.

V. Conclusion

Based on the description, on III and IV point, proved that trigonometric Poschl Teller potential plus Rosen Morse non central potential for group of *shape invariance* potential can be solved with Supersymmetric method.

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