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# Some Inequalities on the Hausdorff Dimension In The Ricker Population Model

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***Abstract :*** *In this paper, we consider the Ricker population model* $f(x) = x\, e^{r(1-\frac{x}{k})}$ *, where r is the control parameter and k is the carrying capacity, and obtain some interesting inequalities on the Hausdorff dimension. Our idea can be extended to Higher dimensional models for further investigation in our research field.*

***Key Words:*** *Hausdorff dimension / Ricker's model / Fractals 2010subject classification : 37 G 15, 37 G 35 ,37 C 45*

## I. Introduction

One natural question with a dynamical system is: how chaotic is a system's chaotic behavior ? To give a quantitative answer to that question , the dimension theory plays a crucial role . Why is dimensionality important ? One possible answer is that the dimensionality of the state space is closely related to dynamics. The dimensionality is important in determining the range of possible dynamical behavior. Similarly, the dimensionality of an attractor tells about the actual long- term dynamics. Among various dimensions, Hausdorff dimension plays a very important role as a fractal dimension and measure of chaos , [1,2,5,6,9].

We first define Hausdorff Measure as follows:

If U is any nonempty subset of n- dimensional Euclidean space, $R^n$ , the diameter of U is defined as $|U| = \sup[|x-y| : x, y \in U]$ , i.e. the greatest distance apart of any pair of points in U . If $\{U_i\}$ is a countable [or finite] collection of sets of distance at most $\delta$ that covers F , i.e. $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 \le |U_i| \le \delta$ for each i , we say that $\{U_i\}$ is a $\delta$ cover of $F \subset R^n$ and $\delta < 1$ ,

Suppose that F is a subset of $R^n$ and s is a nonnegative number. For any $\delta > 0$ , we define

$$H_s^{\delta}(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{ - cover of F }\} \qquad (1.1)$$

As $\delta$ decreases, the class of permissible covers of F in ( 1.1) is reduced. Therefore, the infimum $H_{\delta}^s(F)$ increases, and so approaches a limit as $\delta \to 0$ . We write

$H^s(F) = \lim_{\delta \to 0} H_{\delta}^s(F)$ . We call $H^s(F)$ the s- dimensional Hausdorff Measure of F . It is clear that for any given set F , $H_{\delta}^s(F)$ is non-increasing with s , so $H^s(F)$ is also non- increasing. In fact, if t > s and $\{U_i\}$ is a $\delta$ -cover of F we have

$$\sum_i |U_i|^t \le \sum_i |U_i|^{t-s} |U_i|^s \le \delta^{t-s} \sum_i |U_i|^s$$

So, taking infima , $H_{\delta}^y(F) \le \delta^{t-s} H_{\delta}^s$ . Letting $\delta \to 0$ , we see that $H^s(F) < \infty$ . Thus, $H^s(F) = 0$ for t > s . Hence a graph of $H^s(F)$ against s shows that there is a critical value of s at which $H^s(F)$ jumps from $\infty$ to 0 . This critical value is called the Hausdorff dimension of F and written as $\dim_H(F)$. Formally,

$\mathrm{Dim}_H(F) = \inf\{ s \ge 0 : H^s(F) = 0 \} = \sup\{ s : H^s(F) = \infty \}$ so that

$H^s(F) = \infty$ if $0 \le s < \dim_H(F)$ and 0 if $s > \dim_H(F)$ .

If $s = \dim_H(F)$, then $H^s(F)$ = may be zero or infinite , or may satisfy $0 < H^s(F) < \infty$ .

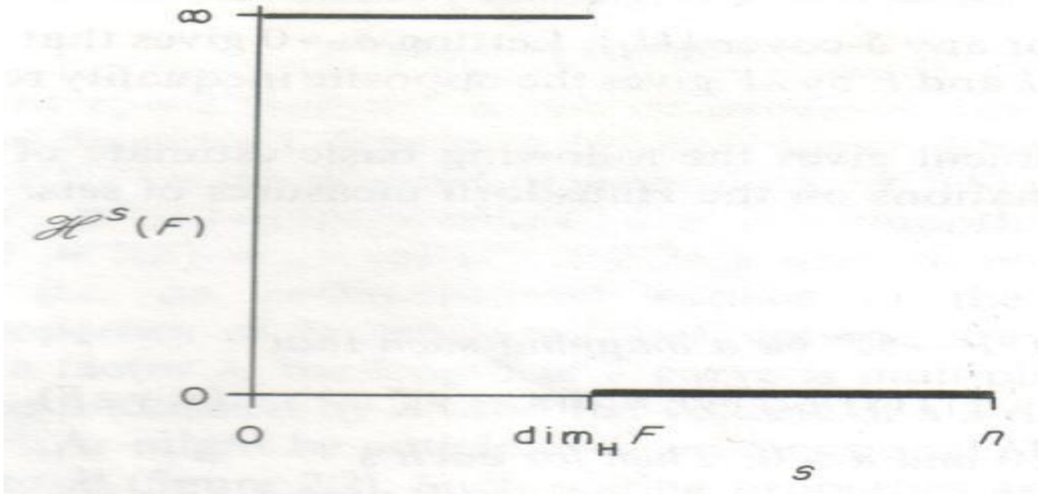


**Fig 1 : Graph of $H^s(F)$ against s for a set F . The Hausdorff dimension is the value of s at which the "jump" from ∞ to 0 occurs .**

Some alternative equivalent definitions for Hausdorff dimension and the method of calculation are available in [6] . It is important to note that most dimension calculations involve an upper estimate and a lower estimate, which are hopefully equal. Each of these estimates usually involves a geometric observation followed by a calculation , [3,6,7,10]

## II. The Main Results:

**CALCULATION OF HAUSDORFF DIMENSION**: [ 5,6,8]

Let us consider the model $f(x) = xe^{r(1-x/k)}$ . We take the parameter value and the constant value in such a way that the function becomes contracting. So as shown in the graph we can take $x_1$ and $x_2$ such that the function becomes

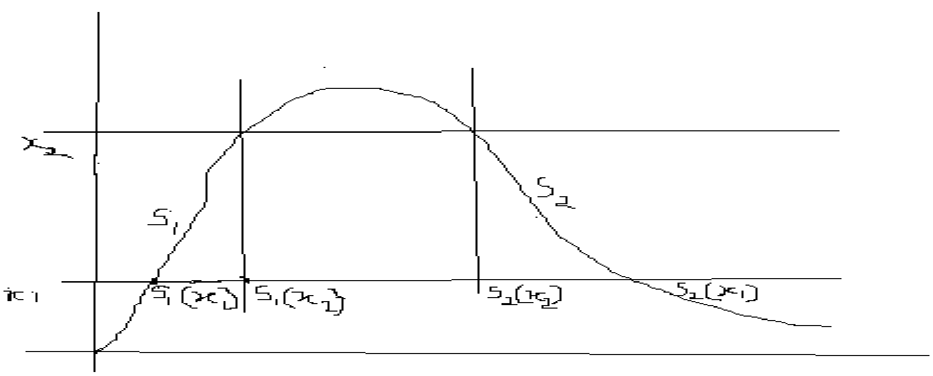


**Fig 2**

$f:[s_1(x_1), s_1(x_2)] \cup [s_2(x_2), s_2(x_1)] \to [x_1, x_2]$. Being a unimodel function, if we consider the inverse of f it will give us two values in that particular range. Let us consider the two inverse functions as $s_1(x)$ and $s_2(x)$ , where $s_1:[x_1,x_2] \to [s_1(x_1), s_1(x_2)]$ and $s_2:[x_1,x_2] \to [s_2(x_2), s_2(x_1)]$. [from the figure 2 we can see the domain and range.]

To get the Hausdroff dimension we use the following two propositions heavily:

- A.Proposition 9.6 in the book fractal geometry by Kennith Falconer,[6] states:

Let F be the attractor of an IFS consists of contractions $\{s_1.s_2,\ldots,s_m\}$ on a closed subset D of $R^n$ such that $|s_i(x) - s_i(y)| \leq c_i|x - y|$ ($x,y \in D$) with $0<c_i<1$ for each i. Then $\dim_H F \leq s$ , where $\sum_{i=1}^{m} c_i^s = 1$

- B.Proposition 9.7 in the book fractal geometry by Kennith Falconer,[] states:

Let F be the attractor of an IFS consists of contractions $\{s_1.s_2,\ldots,s_m\}$ on a closed subset D of $R^n$ such that $|s_i(x) - s_i(y)| \geq b_i|x - y|$ ($x,y \in D$) with $0< b_i <1$ for each i.Then $\dim_H F \geq s$ , where $\sum_{i=1}^{m} b_i^s = 1$

So we shall try to use these two theorems to calculate the Hausdroff dimension.

By their properties we have $f(s_i(x)) = x$ for i =1,2. So we have

$s_i(x).e^{r\left(1-\frac{s_i(x)}{k}\right)} = x$ (1.2) , [as $f(x)=x\, e^{r(1-x/k)}$] for i =1,2. Solving this equation we may get the functions $s_i(x)$. However solving this type of nonlinear equation analytically is not so easy.

We differentiate equation (1.2) w.r.t x and we get

$s_i^{/}(x) = \frac{e^{-r(1-s_i(x)/k)}}{1-\frac{r}{k}s_i(x)}$ ,for i=1,2

If we again differentiate we have

$\frac{d^2 s}{dx^2} = \left(\frac{ds}{dx}\right)^2 \frac{r}{k}\,[1 + \frac{1}{1-rs(x)/k}]\ldots$ ( 1.3)

The maximum or minimum value occurs at s(x)= 2k/r, which is the end point of the definition of our function (if we consider 2k/r as the end point). Clearly $\frac{d^2 s}{dx^2}$ >0 for $s_1(x)$ (as $s_1(x) < k/r$ , we can see this from the above figure). So $s_1^{/}(x)$ is an increasing function. Hence the bounds are given by as follows

$\frac{e^{-r(1-s(x_1)/k)}}{1-r.s(x_1)/k} < \frac{ds_1}{dx} < \frac{e^{-r(1-s(x_2)/k)}}{1-rs(x_2)/k}$ where s means $s_1$.Again we have $x_1>0$ .From this we get

$e^{-r} < \frac{e^{-r(1-s(x_1)/k)}}{1-r.s(x_1)/k} < \frac{ds_1}{dx} < \frac{e^{-r(1-s(x_2)/k)}}{1-rs(x_2)/k}\ \ldots$ ( 1.4)

Again we consider the equation (1.4) for $s = s_2$.

As $k/r<s_2(x)<2k/r$

That is , $1 < r\, s_2(x)/k < 2$

That is, $-1>- r\, s_2(x)/k >- 2$ or, $0>1- r\, s_2(x)/k >- 1$

Hence $\frac{d^2 s}{dx^2} < 0$ for $s_2(x)$ .So $s_2^{/}(x)$ is a decreasing function. Again we see $s_2(x)$ is also decreasing function. Hence the bounds are given by

$\frac{e^{-r(1-s(x_1)/k)}}{1-r.s(x_1)/k} < \frac{ds_2}{dx} < \frac{e^{-r(1-s(x_2)/k)}}{1-rs(x_2)/k}$ (1.5)

Where s means $s_2$.

If 2k/r lies in the range of $s_2$ then the bounds becomes

Max and min of $\{\frac{e^{-r(1-s(x_1)/k)}}{1-r.s(x_1)/k} , \frac{e^{-r(1-s(x_2)/k)}}{1-rs(x_2)/k} ,-e^{(2-r)}\}$ (1.6)

The Hausdorff dimension of our model lies in between the maximum and the minimum values obtained from (1.6) .

## III. Conclusion

We select the upper and lower bound for $\left|\frac{ds}{dx}\right|$ where s means $s_1$ and $s_2$ from propositions (A) and (B), say c1 and c2 then the upper and lower bounds of the Hausdroff dimension will be {a1,a2} where $2.c1^{a1}=1$ and $2.c2^{a2}=1$ . If we are not satisfied with the bounds then we may go for the composite mappings like $s_1s_1, s_1s_2, s_2s_2, s_2s_1$.As $s_1$ is an increasing function so $s_1s_1$, $s_1s_2$. Moreover, as $s_2$ is a decreasing function so is $s_2s_2, s_2s_1$. Again $s_1s_2 :[s_2(x_2),s_2(x_1)] \rightarrow [s_1s_2x_2, s_1s_2x_1]$.For all these we need $[s_1x_1, s_1x_2] \subseteq [x_1,x_2]$ unfortunately these does not happen all the time. But if we take $x_1$ and $x_2$ as the fixed points then there will be no problem, again in our case 0 is one obvious fixed point if we consider 0 then our function will no longer be contracting so we have to see other fixed points other than 0 . So if we go with the fixed points then proceeding with the above procedure exactly in the same way we may further contract the bounds, and hence get a better Hausdroff dimension.

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