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Analytic Solution of Natural Convection Flow of a Non-Newtonian Fluid between Two Vertical Flat Plates Using Homotopy Perturbation Method (HPM)

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Abstract: *An analysis has been performed to study the natural convection of a non-Newtonian fluid between two infinite parallel vertical flat plates and the effects of the non-Newtonian nature of fluid on the heat transfer are studied. The governing boundary layer and temperature equations for this problem are reduced to an ordinary form and are solved by Homotopy perturbation method (HPM) and numerical method. Velocity and temperature profiles are shown graphically. The obtained results are valid for the whole solution domain with high accuracy. These methods can be easily extended to other linear and non-linear equations and so can be found widely applicable in engineering and science.*

Keywords*: Homotopy perturbation method (HPM); natural convection; non-Newtonian fluid; numerical method (NM).*

1. Introduction

Heat transfer by natural convection frequently occurs in many physical problems and engineering applications such as geothermal systems, heat exchangers, chemical catalytic reactors, fiber and granular insulation, packed beds, petroleum reservoirs, and nuclear waste repositories. In review of its importance, the flow of Newtonian and non-Newtonian fluids through two infinite parallel vertical plates has been investigated by numerous authors.

A non-Newtonian fluid is a fluid whose flow properties are not described by a single constant value of viscosity. Many polymer solutions and molten polymers are non-Newtonian fluids, as are many commonly found substances such as ketchup, starch suspensions, paint, blood, and shampoo. In a Newtonian fluid, the relation between the shear stress and the strain rate is linear (and if one were to plot this relationship, it would pass through the origin), the constant of proportionality being the coefficient of viscosity. In a non-Newtonian fluid, the relation between the shear stress and the strain rate is nonlinear, and can even be time dependent. Therefore, a constant coefficient of viscosity cannot be defined. A ratio between shear stress and the rate of strain (or shear-dependent viscosity) can be defined, this concept being more useful for fluids without time-dependent behavior. Although the concept of viscosity is commonly used to characterize a material, it can be inadequate to describe the mechanical behavior of a substance, particularly non-Newtonian fluids. They are best studied through several other rheological properties which relate the relations between the stress and strain rate tensors under many different flow conditions, such as oscillatory shear, or extensional flow which are measured using different devices or rheometers. The properties are better studied using tensor-valued constitutive equations, which are common in the field of continuum mechanics.

The natural convection problem between vertical flat plates for a certain class of non-Newtonian fluids has been carried out by Bruce and Na [1]. Other laminar natural convection problems involving heat transfer have also been studied [2]. However, Rajagopal [3] presented a complete thermodynamic analysis of the constitutive functions. These scientific problems and phenomena are modeled by ordinary or partial differential equations. In most cases, analytical solutions cannot be applied to these problems, so these equations should be solved using special techniques.

In recent years, much attention has been devoted to the newly developed methods to construct an analytic solution of equation; such as the method including the Perturbation techniques. Perturbation techniques are too strongly dependent upon the so-called "small parameters" [4]. Many other different methods have been

Introduced to solve nonlinear equations such as Adomian decomposition method (ADM) [5–8], Hamiltonian Approach [9-11] and Exp-Function method [12, 13].
Homotopy perturbation method is one of the well-known methods to solve non-linear equations that does not need to any small parameter. This method has been introduced by Prof. J. H. He in 1998 [14].
The method has been used by many authors [15–19] in a wide variety of scientific and engineering applications to solve different types of governing differential equations: linear and non-linear, homogeneous and non-homogeneous, and coupled and decoupled as well. This method offers highly accurate successive approximations of the solution. Therefore in the present work we re-examine the natural convection of a non-Newtonian fluid, namely the Rivlin–Ericksen fluid of grade three, between two infinite parallel vertical flat plates, and attempt to obtain its solution using the HPM.

2. Fundamentals of the Homotopy Perturbation Method

To illustrate the basic ideas of this method, we consider the following equation [17-19]:

$$A(u) - f(r) = 0 \qquad r \in \Omega \tag{1}$$

With the boundary condition of:

$$B(u, \frac{\partial u}{\partial n}) = 0 \qquad r \in \Gamma \tag{2}$$

Where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.
A can be divided into two parts which are L and N, where L is linear and N is nonlinear. Eq. (1) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \qquad r \in \Omega \tag{3}$$

Homotopy perturbation structure is shown as follows:

$$H(\nu, p) = (1-p)\,[L(\nu) - L(u_0)] + p[A(\nu) - f(r)\,] = 0 \tag{4}$$

Where:

$$\nu(r, p): \ \Omega \times [0,1] \to R \tag{5}$$

In Eq. (4), $p \in [0,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (4) can be written as a power series in p, as following:

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + p^3\nu_3 + \ldots \tag{6}$$

And the best approximation for solution is:

$$u = \lim_{p \to 1}(\nu_0 + p\nu_1 + p^2\nu_2 + p^3\nu_3 + \ldots) \tag{7}$$

3. Description of the Problem

A schematic of the problem under study is shown in **Fig. 1**. It consists of two flat plates that can be positioned vertically. A non-Newtonian fluid is in two flat plates a distance $2b$ apart. The walls at $x = +b$ and $x = -b$ are held at constant temperatures Θ_2 and Θ_1, respectively, where $\Theta_1 > \Theta_2$.
This difference in temperature causes the fluid near the wall at $x = -b$ to rise and the fluid near the wall at $x = +b$ to fall.

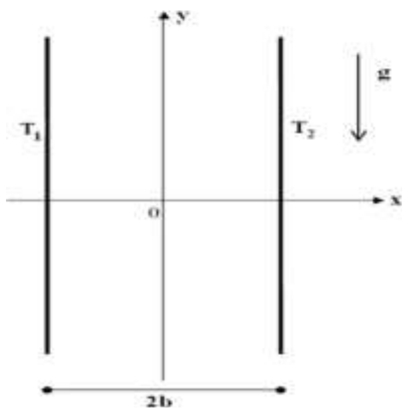

Fig. 1. Schematic diagram of the problem under consideration

The equation of motion is [3]:

$$\mu\frac{d^2V}{dx^2}+6\beta^3\left(\frac{dV}{dx}\right)^2\frac{d^2V}{dx^2}+\rho_0\gamma(\Theta-\Theta_m)g=0 \tag{8}$$

And the energy equation as follows:

$$\mu\left(\frac{dV}{dx}\right)^2+2\beta^3\left(\frac{dV}{dx}\right)^4+k\frac{d^2\Theta}{dx^2}=0 \tag{9}$$

Rajagopal [3] has demonstrated that by using the similarity variables:

$$v=\frac{V}{V_0},\ \eta=\frac{x}{b},\ \theta=\frac{\Theta-\Theta_m}{\Theta_1-\Theta_2} \tag{10}$$

The Navier–Stokes and Energy equations can be reduced to the following pair of ordinary differential equations [3]:

$$\frac{d^2v}{d\eta^2}+6\delta\left(\frac{dv}{d\eta}\right)^2\frac{d^2v}{d\eta^2}+\theta=0 \tag{11}$$

And

$$\frac{d^2\theta}{d\eta^2}+E.\mathrm{Pr}\left(\frac{dv}{d\eta}\right)^2+2\delta E.\mathrm{Pr}\left(\frac{dv}{d\eta}\right)^4=0 \tag{12}$$

Where:

$$E=\frac{V_0^{\ 2}}{c\left(\Theta_1-\Theta_2\right)} \tag{13}$$

And:

$$\mathrm{Pr}=\frac{\mu c}{k} \tag{14}$$

And:

$$\delta=\frac{6\beta^3V_0^{\ 2}}{\mu b^2} \tag{15}$$

Where c is the specific heat of the fluid. The appropriate boundary conditions are:

$$v=0,\ \theta=+\frac{1}{2}\ \text{ at }\ \eta=-1 \tag{16}$$

$$v=0,\ \theta=-\frac{1}{2}\ \text{ at }\ \eta=+1 \tag{17}$$

In the next sections, we shall solve the system of Eqs. (11) and (12) by using the HPM. The equations are coupled and highly nonlinear.

4. Applacation

According to the HPM, we can construct a homotopy of Eq (11) and Eq (12) as follows:

$$H_1(p,\eta)=(1-p)\left[\frac{d^2v}{d\eta^2}+\theta\right]+p\left(\frac{d^2v}{d\eta^2}+6\delta\left(\frac{dv}{d\eta}\right)^2\frac{d^2v}{d\eta^2}+\theta\right) \tag{18}$$

And:

$$H_2(p,\eta)=(1-p)\left[\frac{d^2\theta}{d\eta^2}\right]+p\left[\frac{d^2\theta}{d\eta^2}+E.\mathrm{Pr}\left(\frac{dv}{d\eta}\right)^2+2\delta E.\mathrm{Pr}\left(\frac{dv}{d\eta}\right)^4\right] \tag{19}$$

We can assume that the solution of Eq. (11) and Eq. (12) can be written as a power series in p , as following:

$$v(\eta) = v_0(\eta) + pv_1(\eta) + p^2 v_2(\eta) + p^3 v_3(\eta) + \ldots \tag{20}$$

And:

$$\theta(\eta) = \theta_0(\eta) + p\theta_1(\eta) + p^2\theta_2(\eta) + p^3\theta_3(\eta) + \ldots \tag{21}$$

Substituting Eqs. (20) and (21) into Eq. (11) and (12) and arranging the coefficients of ‘‘ p ’’ powers, we have:

$$\begin{aligned}&\left[\theta_0(\eta) + \frac{d^2}{d\eta^2}v_0(\eta)\right]p^0 + \\ &\left[\theta_1(\eta) + \frac{d^2}{d\eta^2}v_1(\eta) + 6\delta\left(\frac{d}{d\eta}v_0(\eta)\right)^2\left(\frac{d^2}{d\eta^2}v_0(\eta)\right)\right]p^1 + \\ &\left[12\delta\left(\frac{d}{d\eta}v_0(\eta)\right)\left(\frac{d}{d\eta}v_1(\eta)\right)\left(\frac{d^2}{d\eta^2}v_0(\eta)\right) + \theta_2(\eta) + 6\delta\left(\frac{d}{d\eta}v_0(\eta)\right)^2\left(\frac{d^2}{d\eta^2}v_1(\eta)\right) + \frac{d^2}{d\eta^2}v_2(\eta)\right]p^2 + \ldots\end{aligned} \tag{22}$$

And

$$\begin{aligned}&\left[\frac{d^2}{d\eta^2}\theta_0(\eta)\right]p^0 + \\ &\left[\left(\frac{d^2}{d\eta^2}\theta_1(\eta)\right) + E.\Pr\left(\frac{d}{d\eta}v_0(\eta)\right)^2 + 2\delta E.\Pr\left(\frac{d}{d\eta}v_0(\eta)\right)^4 + \left(\frac{d^2}{d\eta^2}\theta_0(\eta)\right)\right]p^1 + \\ &\left[8\delta E.\Pr\left(\frac{d}{d\eta}v_0(\eta)\right)^3\left(\frac{d}{d\eta}v_1(\eta)\right) + 2E.\Pr\left(\frac{d}{d\eta}v_0(\eta)\right)\left(\frac{d}{d\eta}v_1(\eta)\right) + \left(\frac{d^2}{d\eta^2}\theta_2(\eta)\right)\right]p^2 + \ldots\end{aligned} \tag{23}$$

To determine $\theta_0(\eta)$, the coefficient of p^0 in Eq. (23) must be zero:

$$p^0 : \left[\frac{d^2}{d\eta^2}\theta_0(\eta)\right] = 0 \tag{24}$$

With the boundary condition of:

$$\theta_0 = -\frac{1}{2} \quad \text{at} \quad \eta = +1 \tag{25}$$

$$\theta_0 = +\frac{1}{2} \quad \text{at} \quad \eta = -1$$

Busing the boundary condition we have:

$$\theta_0(\eta) = -\frac{1}{2}\eta \tag{26}$$

To determine $v_0(\eta)$, the coefficient of p^0 in Eq. (22) must be zero:

$$p^0 : \left[\theta_0(\eta) + \frac{d^2}{d\eta^2}v_0(\eta)\right] = 0 \tag{27}$$

With the boundary condition of:

$$v_0 = 0 \quad \text{at} \quad \eta = +1 \tag{28}$$

$$v_0 = 0 \quad \text{at} \quad \eta = -1$$

Using the boundary condition we have:

$$v_0(\eta) = \frac{1}{12}\eta^3 - \frac{1}{12}\eta \tag{29}$$

Now using the coefficient of p^1 in Eq. (23), $v_0(\eta)$ and $\theta_0(\eta)$, $\theta_1(\eta)$ can be obtained. Also the boundary condition is:

$\theta_1 = 0$ at $\eta = +1$ (30)

$\theta_1 = 0$ at $\eta = -1$

$$\theta_1(\eta) = -\frac{1}{10368} E.\Pr\left(\frac{9}{10}\delta\eta^{10} - \frac{27}{14}\delta\eta^8 + \frac{1}{30}(54\delta + 648)\eta^6 + \frac{1}{12}(-12\delta - 432)\right. \quad (31)$$

$$+\frac{19}{725760}\delta E.\Pr + \frac{1}{480} E\Pr$$

And using the coefficient of p^1 in Eq. (22), $v_0(\eta)$, $\theta_0(\eta)$ and $\theta_1(\eta)$, $v_1(\eta)$ can be obtained. Also the boundary condition is:

$v_1 = 0$ at $\eta = +1$ (32)

$v_1 = 0$ at $\eta = -1$

$$v_1(\eta) = \frac{1}{1520640}\delta E.\Pr\eta^{12} - \frac{1}{483840}E.\Pr\delta\eta^{10} + \frac{1}{26880}E.\Pr\eta^8 - \frac{1}{224}\delta\eta^7 - \frac{}{31} \quad (33)$$

$$\frac{1}{8640}E.\Pr\eta^6 + \frac{1}{160}\delta\eta^5 - \frac{1}{248832}\delta E.\Pr\eta^4 + \frac{1}{3456}E.\Pr\eta^4 - \frac{1}{288}\delta\eta^3 - \frac{19}{145152}$$

$$\frac{1}{960}E.\Pr\eta^2 + \frac{17}{10080}\delta\eta + \frac{29}{2737152}\delta E.\Pr + \frac{67}{80640}E.\Pr$$

Similary, using the coefficient of p^2 in Eq. (23), $\theta_2(\eta)$ and the coefficient of p^2 in Eq. (22), $v_2(\eta)$ can be obtained. The solutions $v_2(\eta)$ and $\theta_2(\eta)$ were too long to be mentioned here, therefore, they are shown graphically also the results are tabulated in Table 1. If we add one more term to the power series in p (Eq. (6)) we can easily obtain the fourth -order approximation. Upon, solving the problem, discussed above, we obtain an astonishingly accurate solution for the third-order approximations. The third-order solution of the problem when $p \to 1$ will be as follows:

$$v(\eta) = v_0(\eta) + pv_1(\eta) + p^2 v_2(\eta) \quad (34)$$

$$\theta(\eta) = \theta_0(\eta) + p\theta_1(\eta) + p^2\theta_2(\eta) \quad (35)$$

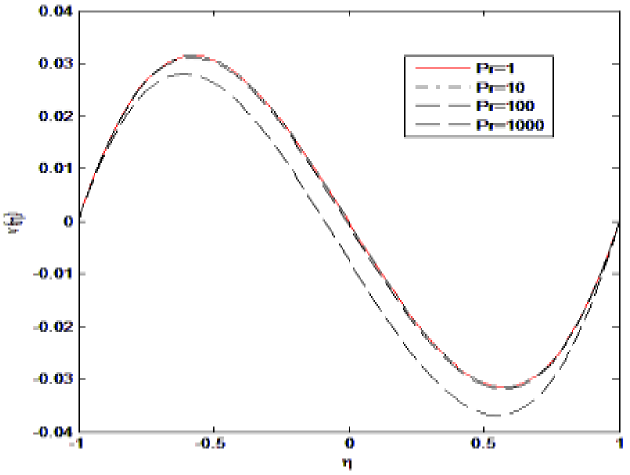

Fig. 2. Results of $v(\eta)$ for various Pr when $\delta = 1$, E=1.

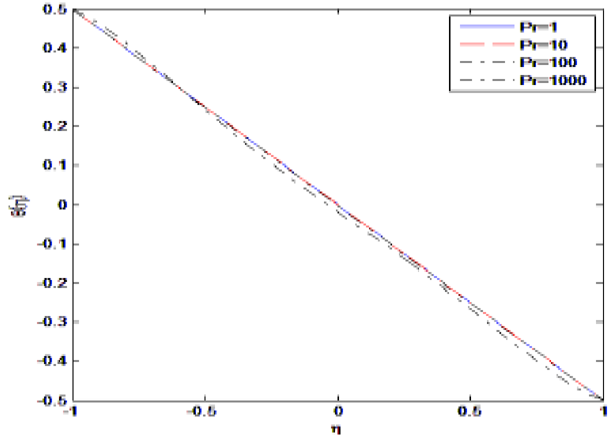

Fig. 3. Results of $\theta(\eta)$ for various Pr when $\delta=1,\ \mathrm{E}=1.$

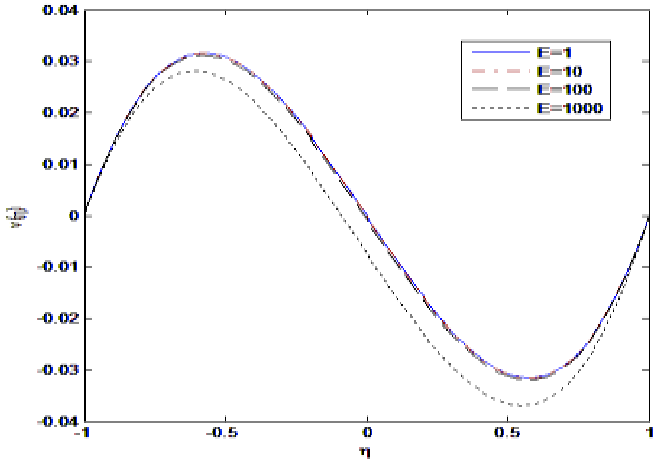

Fig. 4. Results of $v(\eta)$ for various E when $\delta=1,\ \mathrm{Pr}=1.$

Table 1. The results of HPM and NM methods for $v(\eta)$ and $\theta(\eta)$ when $\delta=0.5,\ \mathrm{Pr}=1,\ \mathrm{E}=1.$

	$v(\eta)$			$\theta(\eta)$		
η	HPM	Numerical	Error (%)	HPM	Numerical	Error (%)
-1.0	0.0000	0.0000	0.0000	0.5000	0.5000	0.0000
-0.8	0.0239	0.0231	3.4632	0.4007	0.4007	0.0000
-0.6	0.0322	0.0314	2.5478	0.3012	0.3011	0.0332
-0.4	0.0284	0.0277	2.5271	0.2016	0.2015	0.0496
-0.2	0.0166	0.0167	0.5988	0.1019	0.1018	0.0982
0	0.00080	0.00075	6.6667	0.0021	0.00200	5.0000
0.2	-0.0151	-0.0147	2.7211	-0.0981	-0.0981	0.0000
0.4	-0.0271	-0.0265	2.2642	-0.1984	-0.1984	0.0000
0.6	-0.0312	-0.0305	2.2436	-0.2988	-0.2989	0.0335
0.8	-0.0234	-0.0226	3.5398	-0.3993	-0.3993	0.0000
1.0	0.0000	0.0000	0.0000	-0.5000	-0.5000	0.0000

The results of different methods of HPM, and Numerical are compared in **Table 1**. **Fig. 2** shows the result of $v(\eta)$, for various Prandtl number Pr when $\delta=1,\ \mathrm{E}=1$, **Fig. 3** shows the result of $\theta(\eta)$, for various Prandtl number Pr when $\delta=1,\ \mathrm{E}=1$ and **Fig. 4** shows the results of $v(\eta)$ for various E when $\delta=1,\ \mathrm{Pr}=1.$

5. Conclusion

In the present work, we have applied the HPM to compute the natural convection of an incompressible fluid of grade three between two infinite parallel vertical plates. The figures and tables clearly show that the results by HPM are in excellent agreement with the results of Numerical solution. According to **Figs. 2** and **3**, by increasing Pr the nondimensional velocity $v(\eta)$ and temperature $\theta(\eta)$ are increased respectively. By increasing E in **Fig. 4**, the non dimensional velocity $v(\eta)$ is increased. According to the **Table 1** this approximate analytical solution is in excellent agreement with the corresponding numerical solutions.
HPM provides highly accurate numerical solutions in comparison with other methods. The HPM does not need a small parameter. Finally, it has been attempted to show the capabilities and the wide-range applications of the Homotopy perturbation method in comparison with the numerical solution of natural convection flow of a non-Newtonian fluid between two vertical flat plat problems.
The solutions are quite elegant and fully acceptable in accuracy. Governing equations are easily solved by the analytical method. Consequently, these equations are solved by the numerical method using MAPLE 10, mathematical software, whose results are given in following figures and table.

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