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New Kronecker product decompositions and its applications

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Abstract: *Firstly, two new kinds of Kronecker decompositions is developed, i.e. KPGD and KPID; Secondly, the sufficient and necessary conditions and algorithms of Kronecker product(KPD), KPGD, and KPID are discussed; At last, some useful properties of the rank of the sum of Kronecker product gemel decompositions are obtained.*

Keywords: *Kronecker product decomposition, gemel decomposition, isomer decomposition, rank, dimensionality reduction.* ***MSC:*** *65L09,65Y04,15A69*

I. Introduction

Because of its elegant algebraic properties, the Kronecker product is a useful tool to solve matrix equations and the nearest kronecker product problems[1,2], do inference in multivariate analysis[3], and construct fast and practical algorithms in signal processing, image processing, computer vision, semidefinite programming, quantum computing, linear systems and stochastic automata networks etc. see[1,4,5,6,7,8,9,10] and so on. Meantime, the applications in nearly all those areas are related to some certain kinds of Kronecker product decompositions which in fact are the inverse problems of Kronecker product, see [3,13,14,18] etc.

Usually, Kronecker product decomposition(KPD) means that a matrix$M$ can be transformed to the ronecker product form of the other matrices $A;B$, *i:e:* $M = A \otimes B$; Kronecker product gemel decomposition(KPGD) means the Kronecker product with the special case $A = B$; And Kronecker product isomer decomposition(KPID) corresponds to the case $M = A \otimes A'$. Obviously, these decompositions often have many solutions as well as the other inverse problems. In this direction, Eugene Tyrtyshnikov[14] has some interesting work about Kronecker ranks; T.G.Kolda[12] has some meaningful work on orthogonal tensor decompositions; DE Launey and Seberry[18] developed some properties and their applications on the strong Kronecker product; In addition, Sadegh Jokar and Molker Mehrmann[11], Jun-e Feng, James Lam, Yimin Wei[13] etc, have obtained some useful properties of the sum of Kronecker products. Undoubtedly,
these different decompositions are helpful for dimensionality reduction procedure which is very important key for high dimensional image processing and gene analyzing.

Unfortunately, many natural questions about seemingly “simple”cases are still not answered in spite of an ever increasing interest and some significant results with applications, such as the conditions and the rank of the sum of these decompositions. In this paper, from the perspective of the inverse problem theory, we mainly explore the sufficient and necessary conditions and algorithms of KPD, KPGD, KPID, and obtain some useful properties of the rank of the sum of Kronecker product gemel decomposition. And our research works are mainly motivated by doing multivariate statistical inference and huge dimensional statistical analysis, solving high dimensional matrix equations and constructing the algorithm of image processing and computer vision.

II. KPD, KPGD And KPID Problems

Obviously, the essential preconditions of KPD($M = A \otimes B$), KPGD($M = A \otimes A$) and KPID($M = A \otimes A'$) means that the matrices M,A,B have the proper columns and rows. And this is easily verified, so, we assume that all the matrices in the following discussion have the suitable column and row numbers.

2.1 The sufficient and necessary conditions of KPD

Let $A=(a_1,\cdots,a_m)\in R^{n\times m}$ with $a_i \in R^n, 1\le i\le m$, then denote $vec(A)=(a_1{}',a_2{}',\cdots,a_m{}')'$.

Firstly, we explore the sufficient and necessary conditions of Kronecker product decomposition, and give the elegant form of this result as follows.

Theorem 2.1. (**KPD**) For an arbitrary matrix $M \in R^{mr\times ns}$, $M = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \cdots & \cdots & \cdots \\ \mathrm{M}_{m1} & \cdots & M_{mn} \end{pmatrix}$, $(\forall M_{ij} \in R^{r\times s}, i=1,\cdots,m, j=1,\cdots,n)$, can be decomposed to the form

$M = A\otimes B$, where $A\in R^{m\times n}$, $B\in R^{r\times s}$, m, n, r, s are some certain integers.

$\Leftrightarrow$ (equivalent to) rank $\{vec(M_{11}), vec(M_{12}),\cdots,vec(M_{1n}),\cdots,vec(M_{mn})\}$ =1.

Miraculously, the proof of this theorem is not difficult, so the details of the proof are omitted.

Remark 1. Generally speaking, the KPD of an arbitrary matrix is not unique, because of (kA) $\otimes$ (lB) = A $\otimes$ B with kl = 1 for arbitrary constants k; l and matrices A,B.

The following algorithm describes the general program of KPD problem that includes whether a matrix can be decomposed or not and how to get the results of KPD.

Algorithm 1(KPD):

step 1: input M, m, n, r, s, verify the size of M is $mr\times ns$ and M! = 0

step 2: define M_{ij}, $i=1,\cdots,m$, $j=1,\cdots,n$

step 3: calculate $vec(M_{ij})$

step 4: if rank{ $vec(M_{11}), vec(M_{12}),\cdots,vec(M_{1n}),\cdots,vec(M_{mn})$ } == 1 goto step 5

else output "can not decomposition"; end

step 5: look for the first M_{ij} ! = 0, define $B = M_{ij}$

step 6: calculate $a_{ij} : vec(M_{ij})$ = a_{ij} $vec(B)$, $i=1,\cdots,m$, $j=1,\cdots,n$

step 7: define $A=(a_{ij})$, $i=1,\cdots,m$, $j=1,\cdots,n$; output A,B; end

2.2 The sufficient and necessary conditions of KPGD and KPID

With the similar inference, we can have the following conditions about KPGD.

Theorem 2.2. (**KPGD**) For an arbitrary matrix $M \in R^{p^2 \times q^2}$ ($M \neq 0$), $M = \begin{pmatrix} M_{11} & \cdots & M_{1q} \\ \cdots & \cdots & \cdots \\ \mathrm{M}_{\mathrm{p1}} & \cdots & M_{pq} \end{pmatrix}$, $(\forall M_{ij} \in R^{p \times q}, i = 1, \cdots, p, j = 1, \cdots, q)$, can be decomposed to the form

$M = A \otimes A$, where $A \in R^{p \times q}$, p, q are some certain integers.

$\Leftrightarrow$ (equivalent to) there exists a subblock $M_{ij} \neq 0$, and $m_{i,j}^{ij} > 0$ where $M_{ij} = (m_{s,t}^{ij})_{s=1,\cdots,p,t=1,\cdots,q}$,

if denote $M_{ij} / \sqrt{m_{i,j}^{ij}} = (a_{kl})_{k=1,\cdots,p,l=1,\cdots,q} = A$ or $-A$, then for $\forall k, l, M_{kl} = a_{kl} A$.

Corollary 2.3. Denote $M \in R^{p^2 \times q^2}$ ($M \neq 0$), $M = (M_{ij}), i = 1, \cdots, p, j = 1, \cdots, q$, $\forall M_{ij} \in R^{p \times q}$,

and $M_{ij} = (m_{s,t}^{ij})_{s=1,\cdots,p,t=1,\cdots,q}$, then the matrix M can be carried out KPGD $(M = A \otimes A) \Rightarrow$

rank $\left\{ vec(M_{11}), vec(M_{12}), \cdots, vec(M_{1q}), \cdots, vec(M_{pq}) \right\}$ =1.

The following algorithm describes the general program of KPGD problem that includes whether a matrix can be decomposed or not and how to get the results of KPGD.

Algorithm 2(KPGD):

step 1: input M, p, q, verify the size of M is $p^2 \times q^2$ and M ! = 0

step 2: define flag=0, M_{ij}, $i = 1, \cdots, p, j = 1, \cdots, q$

step 3: for (i, j), $i = 1, \cdots, p, j = 1, \cdots, q$

if: M_{ij} == 0, continue;

else if: M_{ij} ! = 0&& $m_{i,j}^{ij} \leq 0$ flag=1; break;

else: define $B = M_{ij} / \sqrt{m_{i,j}^{ij}}$ flag=2; break;

step 4: if flag==2&&M == $B \otimes B$ A = B; output A; end

else output "can not gemel decomposition"; end

Theorem 2.4. (**KPID**) For an arbitrary matrix $M \in R^{p^2 \times q^2}$ ($M \neq 0$), $M = \begin{pmatrix} M_{11} & \cdots & M_{1q} \\ \cdots & \cdots & \cdots \\ \mathrm{M}_{\mathrm{p1}} & \cdots & M_{pq} \end{pmatrix}$, $(\forall M_{ij} \in R^{p \times q}, i = 1, \cdots, p, j = 1, \cdots, q)$, can be decomposed to the form

$M = A \otimes A'$, where $A \in R^{p\times q}$, p, q are some certain integers.

$\Leftrightarrow$(equivalent to) there exists a subblock $M_{ij} \neq 0$, and $m_{j,i}^{ij} > 0$ where $M_{ij} = (m_{s,t}^{ij})_{s=1,\cdots,p,t=1,\cdots,q}$, such that $\forall k,l, M_{kl} = a_{kl}A'$, where $A = (a_{kl})_{k=1,\cdots,p,l=1,\cdots,q}$, and $A' = M_{ij} / \sqrt{m_{i,j}^{ij}}$ or $A' = -M_{ij} / \sqrt{m_{i,j}^{ij}}$.

$i = 1,\cdots,k,\ j = 1,\cdots,n$, then

III. The Rank Of The Sum Of KPGD

In this section, we discuss some properties of the rank of the sum of KPGD($\sum_{i=1}^{k} A_i \otimes A_i$).

Lemma 3.1. Let *A* and *B* be n × n real symmetric matrices.
(i) There exists a real orthogonal matrix *Q* such that *Q'AQ* and *Q'BQ* are both diagonal if and only if *AB = BA* (that is *AB* is symmetric).
(ii) The previous result holds for more than two matrices. A set of real symmetric matrices are simultaneously diagonalizable by the same orthogonal matrix *Q* if and only if they commute pairwise.
see George A. F. Seber[17] for more details about matrices simultaneous diagonalization.

Theorem 3.2. Let $\mathrm{A}_1,\cdots, \mathrm{A}_k$ ($k \geq 2$) be $n\times n$ real symmetric and positive definite(> 0) or negative definite(< 0) matrices. If they commute pairwise, then *rank*($\sum_{i=1}^{k} A_i \otimes A_i$) $= \left[rank(A_1)\right]^2 = n^2$.

Proof: By the result of Lemma 3.1, there exists an orthogonal matrix *Q*, such that

$QA_iQ' = diag(\lambda_{i1},\cdots,\lambda_{in})$, $i = 1,\cdots,k$, where $\lambda_{i1},\cdots,\lambda_{in}$ are the eigenvalues of A_i, and $\lambda_{ij} > 0$,

$$(Q\otimes Q)(\sum_{i=1}^{k} A_i \otimes A_i)(Q\otimes Q) = \sum_{i=1}^{k} diag(\lambda_{i1},\cdots,\lambda_{in}) \otimes diag(\lambda_{i1},\cdots,\lambda_{in})$$

Obviously, the $rank(\sum_{i=1}^{k} diag(\lambda_{i1},\cdots,\lambda_{in}) \otimes diag(\lambda_{i1},\cdots,\lambda_{in})) = n^2$, then the proof is completed.

With similar discussions, we have the following two theorems.

Theorem 3.3. Let $\mathrm{A}_1,\cdots, \mathrm{A}_k$ ($k \geq 2$) be $n\times n$ real symmetric matrices, there at least a positive definite(> 0) or negative definite(< 0) matrices. If they commute pairwise, then *rank*($\sum_{i=1}^{k} A_i \otimes A_i$) $= \left[rank(A_1)\right]^2 = n^2$.

Theorem 3.4. Let $\mathrm{A}_1,\cdots, \mathrm{A}_k$ ($k \geq 2$) be $n\times n$ real symmetric and positive definite(> 0) or negative definite(< 0) matrices. If they commute pairwise, then *rank*($\sum_{i=1}^{k} A_i \otimes A_i$) $\geq \max\left\{rank(A_1 \otimes A_1),\cdots, rank(A_k \otimes A_k)\right\}$.

These results will be helpful to study the solutions of the following general Sylvester matrix equation problem[19, 20]:

$$A_1 X A_1 + \cdots + A_k X A_k = C \Leftrightarrow \left(\sum_{i=1}^{k} A_i \otimes A_i\right) vec(X) = vec(C)$$

Theorem 3.5. Let A, B be $n \times n$ real symmetric matrices with $AB = BA$, the eigenvalues of A are $\lambda_1, \cdots, \lambda_n$, then the eigenvalues of B are $\mu_1, \cdots, \mu_n$, define the vector $\lambda = (\lambda_1, \cdots, \lambda_n)'$, $\mu = (\mu_1, \cdots, \mu_n)'$, $\alpha = (\alpha_1, \cdots, \alpha_n)'$, where $\alpha_1, \cdots, \alpha_n$ is an arbitrary permutation of the elements $\mu = (\mu_1, \cdots, \mu_n)'$, $Q = \left\{(i, j) : \lambda_i \lambda_j + \alpha_i \alpha_j = 0, \forall i, j = 1, \cdots, n\right\}, \tau =$ the number of Q, then

$$n^2 - \tau_{\max} \le rank(A \otimes A + B \otimes B) \le n^2 - \tau_{\min}$$

Proof: There exists an orthogonal matrix Q, such that $QAQ' = diag(\lambda_1, \cdots, \lambda_n)$ and $QBQ' = diag(\alpha_1, \cdots, \alpha_n)$. Thus,

$$(Q \otimes Q)(A \otimes A + B \otimes B)(Q' \otimes Q') = diag(\lambda_1, \cdots, \lambda_n) \otimes diag(\lambda_1, \cdots, \lambda_n)$$

$$+ diag(\alpha_1, \cdots, \alpha_n) \otimes diag(\alpha_1, \cdots, \alpha_n)$$

. Calculate the number of nonzero elements of $\left\{\lambda_i \lambda_j + \alpha_i \alpha_j, \forall i, j = 1, \cdots, n\right\}$, and the result is obtained.

IV. Discussion

In this paper, we discuss the sufficient and necessary conditions and algorithms of KPD, KPGD and KPID problems, which play a great role in all kinds of Kronecker product application areas, and obtain some useful properties of the rank of the sum of KPGD($\sum_{i=1}^{k} A_i \otimes A_i$) in simultaneous diagonalization situation which performs some wonderful algebra advantages. More interesting work in the future maybe include the following aspects: the conditions that a matrix can be decomposed to the form $\sum_{i=1}^{k} A_i \otimes B_i$ which is a meaningful work especially in sparse matrices cases[11, 21], decomposing program, and the properties of the rank of the sum of decomposition in more general cases. Also, those Kronecker product decomposition properties maybe associated with seeking the sufficient and necessary conditions under which the following matrix equation $\sum_{i=1}^{n} A_i (I \otimes X) B_i = C$ has a uniquely solution, where A_i, B_i and C_i are known matrices, and X is an unknown matrix.

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