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Applications of Optimization Methods in Finding the Overall Average in Examination Marks

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Abstract

We present some refinements for finding the averages in examination marks. In many high-level examinations, students take different courses from different disciplines. The results are prepared from the marks of different courses of their study without considering the marks have received through a fair process. In this paper, different methods are considered for finding the averages of the subjects taken by the candidates. We only consider the averages of the marks and give them the grades.

Assume that N candidates take q papers from a total number of n papers. Let m_{ij} be the mark scored by candidate i on paper j and each paper is assumed to have equal weighting in the overall assessment. To harmonize standards across all papers, an adjustment parameter p_j (multiplicative) for papers j is introduced along with an 'average' assessment a_i for candidate i. The parameters a_i and p_j are calculated by minimizing a loss function which represents the disagreement between the actual marks and proposed average assessments.

In MATLAB, some defined routines are available for minimization of constrained optimization problem (*fminsearch, fmincon etc*) to compute the parameters, but here we used own MATLAB programs using Fletcher-Reeves (FR) method for the raw data set of examination marks table-1.

Keywords: Fletcher-Reeves, parameter computations; Broyden's and Biggin's scaling of examination marks.

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I. Introduction

In this paper we are interested in measuring the overall average of a student or, in a general to check the ability of a student, so that a weak performance in one paper may be compensated by a strong performance in another paper.

Examinations usually consist of several components. We are interested to find a fair and harmonious way of deriving overall marks from a set of components. Here, we assume that the component marks have been received through a fair process and the only problem that we will apply, is that of combining them fairly and consistently.

One method that is often used is that of simply adding the component marks together to get the overall mark. This assumes that the components are all equally important and are also treated equally. For combining these courses, the candidates may have some core courses, optional courses, or some specialties in pure, Applied, Computational Mathematics, Statistics or studying several special types.

The result is a set of assessments in which not every candidate takes every paper. Yet for the purpose of ranking the candidates and classifying their degrees, a single overall 'average' mark must be assigned to each candidate. It is assumed that each of N candidates take q papers which are selected from a total numbers of n papers. The mark scored by candidate i on paper j is m_{ij} and each paper is assumed to have equal weighting in the overall assessment. Of course, m_{ij} only exists for certain pairs. The papers will vary in their intrinsic difficulty and the examiners in their generosity. The overall ability can be regarded as some function of the component marks in the individual topics.

To harmonize standards across all papers, an adjustment parameter p_j (multiplicative) for paper j is introduced, along with an 'average' measure a_i for candidate i. These parameters are calculated by minimizing a loss function which represents the disagreement between the scaled marks and the ability of the candidates. This idea was used by [6] in a somewhat different context. Murgatroyd ([6, 8]) also followed this philosophy although he mainly considered additive adjustments.

II. Computation of the parameters

2.1 Broyden's Method

A simple form the loss function which treats all candidates and all papers on the same basis is the one proposed by [1, 4],

$$S = \sum_i \sum_j (p_j m_{ij} - a_i)^2 \quad (2.1.1)$$

We choose P_j and a_i to minimize this loss function S. Unfortunately, the solution to this problem is $p_j = 0$, and $a_i = 0$ for all i and j. Thus following [1, 4], we considered a constraint that the total of the marks remains the same after multiplying the adjustment factor p_j. Thus, we have

$$\sum_i \sum_j m_{ij} = \sum_i \sum_j p_j\, m_{ij} \quad (2.1.2)$$

Using (1) and (2) we can construct the Lagrangian

$$L = \sum_i \sum_j (p_j m_{ij} - a_i)^2 - 2\lambda \sum_i \sum_j m_{ij}(p_j - 1.0) \quad (2.1.3)$$

This leads to the equations $a_i = \sum_j \frac{m_{ij} p_j}{n_i}$ (2.1.4)

and $p_j = \frac{\sum_i a_i m_{ij} + \lambda \sum_i m_{ij}}{\sum_i m_{ij}^2}$ (2.1.5)

Where n_i is the number of papers taken by candidate i.

It is very important to state that there may be 100-200 candidates each taking 8 (say) options from given 30-60 papers.

The adjustment factor p_j $(j = 1, \ldots, n)$ for each paper and a_i $(i = 1, \ldots, N)$ for each candidate it would not be unusual to have a constrained optimization problem higher number of variables. For this reason, we normally preferred Fletcher-Reeves optimization method [5], to obtain the solution to the problem which uses function values and gradient only at each iteration, posed by [8,9].

Whereas the variable metric methods such as Davidon-Fletcher-Powell (DFP) or Broyden-Fletcher-Goldfarb-Shanno (BFGS), need the Hessian matrix, which gets updated at each iteration. Fletcher-Reeves methos [5], is an iterative method which searches along a set of mutually conjugate directions. To construct the next search direction, we need the current gradient g_{k+1} and the last search direction d_k (which is stored),

$$d_{k+1} = -g_{k+1} + \alpha_k d_k$$

Where $\alpha_k = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}$

and d_{k+1} is the current search direction.

Since the constraint can be readily used to eliminate one of the variables (p_n say). We have

$$S = \sum_i \sum_j (p_j m_{ij} - a_i)^2 \text{ with } \ p_n = \frac{T}{T_n} - \sum_{j=1}^{n-1} p_j \frac{T_j}{T_n} \tag{2.1.6}$$

where $T_j = \sum_i m_{ij} \equiv$Total marks for paper j, and $T = \sum_j \sum_i m_{ij} \equiv$Total of all marks.

In Fletcher-Reeves method [5], we used first partial derivatives,

$$\frac{\partial S}{\partial a_i} = -2\sum_j (p_j m_{ij} - a_i) \text{ for } i = 1,2,\dots,N \tag{2.1.7}$$

$$\frac{\partial S}{\partial p_j} = \sum_i 2m_{ij}(p_j m_{ij} - a_i) + \sum_i 2m_{in}(p_n m_{in} - a_i)\left(-\frac{T_j}{T_n}\right)$$

$$\text{for } j = 1,2,\dots,n-1. \tag{2.1.8}$$

The results for average marks a_i and the adjustment factors p_j for the examination marks table-1, using Fletcher-Reeves optimization method, we found the optimal values of the variables a_i and p_j ; and these optimal values are presented in column (b) of the Table-2 and Table-3 respectively.

2.2 Some Refinements in the Loss Function using Broyden's Method (1)

It is supposed that the adjustment factor p_j should be close to one. To ensure this, we modify the loss function [1,2], with a penalty component c,

$$S_1 = \sum_i \sum_j (p_j m_{ij} - a_i)^2 + c\sum_j (p_j - 1)^2 \tag{2.2.1}$$

Here if $c = 0,$ then we get the same as above eq. (2.1.1) or eq. (2.1.6) and if $c \to \infty$, $p_j = 1$ and we just calculate the straight average of each candidate as usual.

Now the Lagrangian function for eq. (2.2.1) and the constraint eq. (2.1.2), we get,

$$L_1 = \sum_i \sum_j (p_j m_{ij} - a_i)^2 + c\sum_j (p_j - 1.0)^2 + 2\lambda \sum_i \sum_j m_{ij}(p_j - 1.0) \tag{2.2.2}$$

This leads to the equations $a_i = \sum_j \frac{m_{ij} p_j}{n_i}$ (2.2.3)

and
$$p_j = \frac{\sum_i a_i m_{ij} + \lambda \sum_i m_{ij} + c}{\sum_i m_{ij}^2 + c} \tag{2.2.4}$$

The constraint can be readily used to eliminate one of the variables (p_n say). We have

$$S_1 = \sum_i \sum_j (p_j m_{ij} - a_i)^2$$

With
$$p_n = \frac{T}{T_n} - \sum_{j=1}^{n-1} p_j \frac{T_j}{T_n} \tag{2.2.5}$$

where $T_j = \sum_i m_{ij} \equiv$Total marks for paper j, and $T = \sum_j \sum_i m_{ij} \equiv$Total of all marks.

In Fletcher-Reeves method [5], we used first partial derivatives,

$$\frac{\partial S_1}{\partial a_i} = -2\sum_j (p_j m_{ij} - a_i) \text{ for } i = 1,2,\dots,N \tag{2.2.6}$$

$$\frac{\partial S_1}{\partial p_j} = \sum_i 2m_{ij}(p_j m_{ij} - a_i) + 2c(p_j - 1.0) + 2\left\{\sum_i m_{in}(p_n m_{in} - a_i) + c(p_n - 1.0)\right\}\left(-\frac{T_j}{T_n}\right)$$

$$\text{for } j = 1,2,\dots,n-1. \tag{2.2.7}$$

The results for average marks a_i and the adjustment factors p_j for the examination marks table(1), using Fletcher-Reeves optimization method, we can find the optimal values of the variables a_i and p_j ; and these optimal values are presented in column (c) of the Table-2 and Table-3 respectively.

2.3 Some Refinements in the Loss Function using Broyden's Method (2)

For an effect of penalty component c, we should assume that c should be m^2, where m is a typical mark ($50 < c < 90$).

Values of around 2500-8000 seem reasonable although our calculations show that the outcome is not too sensitive to the actual values of c.

Biggins [1,2,3] show that if we include a fictitious candidate who score c on each paper so that $a = c\bar{p}$ (where $\bar{p}$ is the average value of p_j) for that candidate.

Then the loss function becomes:

$$S_2 = \sum_i \sum_j (p_j m_{ij} - a_i)^2 + c^2 \sum_j (p_j - \bar{p})^2 \qquad (2.3.1)$$

This leads to the equations $a_i = \sum_j \frac{m_{ij} p_j}{n_i}$ (2.3.2)

and
$$p_j = \frac{\sum_i a_i m_{ij} + \lambda \sum_i m_{ij} + c^2 \bar{p}}{\sum_i m_{ij}^2 + c^2} \qquad (2.3.3)$$

In Fletcher-Reeves method we again used first partial derivatives,

$$\frac{\partial S_2}{\partial a_i} = -2 \sum_j (p_j m_{ij} - a_i) \text{ for } i = 1,2, \dots, N \qquad (2.3.4)$$

$$\frac{\partial S_2}{\partial p_j} = \sum_i 2 m_{ij} (p_j m_{ij} - a_i) + 2c^2 (p_j - \bar{p}) +$$

$$2\{\sum_i m_{in}(p_n m_{in} - a_i) + c^2 (p_n - \bar{p})\} \left(-\frac{T_j}{T_n}\right) \qquad (2.3.5)$$

for $j = 1,2, \dots, n-1$.

The results for average marks a_i and the adjustment factors p_j for the examination marks provided, using Fletcher-Reeves optimization method, we can find the optimal values of the variables a_i and p_j ; and these optimal values are presented in column (d) of the Table-2 and Table-3 respectively.

2.4 Biggins' Alternative Method

To have a non-trivial solution for the loss function S in eq. (2.1.1), we considered a constrained optimization problem for the Broyden's method [4]. To get rid of the constraint eq. (2.1.2), Biggin's [3], suggested the loss function as: $S_3 = \sum_i \sum_j \left(\frac{m_{ij} p_j}{a_i} - 1.0\right)^2$

where the parameters a_i and p_j to be minimized using optimization methods and with this loss function, we again note that, if the matrix of marks is split into two blocks, then the indeterminacy occurs again in each block separately. Thus, in such cases, one constraint is imposed on each block to remove this indeterminacy. Therefore, to overcome this issue, we consider a loss function in the form $L(x) = e^x - x - 1$ with $x = log\left(\frac{mp}{a}\right)$ and L *is* a strictly convex function, then the loss function S_3 becomes,

$$S_3 = \sum_i \sum_j \frac{m_{ij} p_j}{a_i} - \sum_i \sum_j log\left(\frac{m_{ij} p_j}{a_i}\right) - 1.0 \qquad (2.4.1)$$

This leads to the equations $a_i = \sum_j \frac{m_{ij} p_j}{n_i}$ (2.4.2)

and $$1/p_j = \frac{\sum_i m_{ij}/\mathrm{a_i}}{\mathrm{n_j}} \quad (2.4.3)$$

where n_j is the total number of candidates taking paper j.

Let $\beta_j = 1/p_j$ then $\beta_j = \frac{\sum_i m_{ij}/\mathrm{a_i}}{\mathrm{n_j}}$.

The Fletcher-Reeves method required first partial derivatives,

$$\frac{\partial S_3}{\partial a_i} = -\sum_j \left(p_j m_{ij}/a_i^2 - 1/a_i\right) \text{ for } i = 1,2,\ldots,N \quad (2.4.4)$$

$$\frac{\partial S_3}{\partial p_j} = \sum_i \left(m_{ij}/a_i - 1/p_j\right) \text{ for } j = 1,2,\ldots,n. \quad (2.4.5)$$

The results for average marks a_i and the adjustment factors p_j for the examination marks given data in table (1), using Fletcher-Reeves optimization method [5], we can find the optimal values of the variables a_i and p_j ; and these optimal values are presented in column (e) of the Table-2 and Table-3 respectively.

2.5 Biggins Modified Method with a Single Fictitious Candidate

A fictitious candidate c as discussed in the previous section is included and assuming that there is only one block, with $p_1 = 1$ as a constraint throughout the block. So, including a fictitious candidate the loss function can be written in the form

$$S_4 = \sum_i \sum_j L\left(log\left(\frac{m_{ij}\, p_j}{a_i}\right)\right) + c\sum_i \sum_j L\left(log\left(\frac{p_j}{p_1}\right)\right). \quad (2.5.1)$$

Again, we take, $L(x) = e^x - x - 1$ then we have the loss function as

$$S_4 = \sum_i \sum_j \frac{m_{ij}\, p_j}{a_i} - \sum_i \sum_j log\left(\frac{m_{ij}\, p_j}{a_i}\right) - 1.0 + c\left(\sum_j \left(\frac{p_j}{p_1}\right) - \sum_j log\left(\frac{p_j}{p_1}\right) - 1\right) \quad (2.5.2)$$

The Fletcher-Reeves method required first partial derivatives,

$$\frac{\partial S_3}{\partial a_i} = -\sum_j \left(p_j m_{ij}/a_i^2 - 1/a_i\right) \text{ for } i = 1,2,\ldots,N \quad (2.5.3)$$

$$\frac{\partial S_3}{\partial p_j} = \sum_i \left(m_{ij}/a_i - 1/p_j\right) + \sum_i \left(1.0 - 1/p_j\right), \text{ for } j = 1,2,\ldots,n. \quad (2.5.4)$$

The results for average marks a_i and the adjustment factors p_j for the examination marks exam data table (1), using Fletcher-Reeves optimization method [5], we can find the optimal values of the variables a_i and p_j ; and these optimal values are presented in column (f) of the Table-2 and Table-3 respectively.

III. Numerical Results and Comparison of Different Methods

The methods from sections 2.1 to section 2.5 were applied for the examination marks data. There were 25 students, and each student took three papers from the given set of eight papers and the paper one was a compulsory for each candidate.

Table: 1. Sample data of marks for 25 candidates each taking 5 papers from 8 papers.

Candidates	Papers								Average for Student
	1	2	3	4	5	6	7	8	
1	75	-	53	-	-	35	-	-	**54.33**
2	32	-	-	30	-	30	-	-	**30.67**
3	41	34	-	33	-	-	-	-	**36.00**
4	42	67	58	-	-	-	-	-	**55.67**
5	42	-	-	52	-	16	-	-	**36.67**
6	44	51	-	-	-	-	61	-	**52.00**
7	45	54	-	64	-	-	-	-	**54.33**
8	48	51	-	-	40	-	-	-	**46.33**
9	48	43	31	-	-	-	-	-	**40.67**
10	49	36	46	-	-	-	-	-	**43.67**
11	51	54	33	-	-	-	-	-	**46.00**
12	51	50	44	-	-	-	-	-	**48.33**
13	56	60	-	-	-	-	56	-	**57.33**
14	57	-	-	46	-	23	-	-	**42.00**
15	57	69	-	-	81	-	-	-	**69.00**
16	58	-	-	55	-	37	-	-	**50.00**
17	63	-	75	-	61	-	-	-	**66.33**
18	64	77	-	-	57	-	-	-	**66.00**
19	67	44	-	-	-	-	67	-	**59.33**
20	67	72	-	-	-	-	72	-	**70.33**
21	69	67	61	-	-	-	-	-	**65.67**
22	72	-	-	59	-	-	-	35	**55.33**
23	78	67	-	-	66	-	-	-	**70.33**
24	86	84	74	-	-	-	-	-	**81.33**
25	73	-	63	-	-	51	-	-	**62.33**
Average in Papers	**57.4**	**57.6**	**53.8**	**48.4**	**61.0**	**32.0**	**64.0**	**35.0**	

The methods discussed above from section 2.1 to section 2.5, were applied using Fletcher-Reeves optimization method for finding the overall averages a_i are presented in table 2 and the adjustment factors p_j in table 3 for the students in the forms of columns from (a) to (f), respectively.

Table: 2. The averages a_i, for the 25 candidates taking 3 papers out of 8. The columns (a)-(f) correspond to different methods of finding average.

Candidates	(a)	(b)	(c)	(d)	(e)	(f)
1	54.33	60.65	59.52	59.46	61.56	59.96
2	30.67	35.09	33.83	33.67	35.74	33.94
3	36.00	34.15	34.26	34.27	33.97	34.25
4	55.67	55.06	55.66	55.79	55.10	55.71
5	36.67	37.97	37.16	37.02	38.27	37.24
6	52.00	47.81	48.98	49.01	46.94	48.59
7	54.33	51.51	51.60	51.56	51.30	51.46
8	46.33	44.82	45.49	45.56	44.80	45.54
9	40.67	39.81	40.21	40.31	39.73	40.33
10	43.67	43.24	43.63	43.72	43.25	43.78
11	46.00	44.95	45.43	45.54	44.85	45.53
12	48.33	47.58	48.06	48.17	47.55	48.18
13	57.33	53.01	54.18	54.24	52.13	53.86
14	42.00	44.45	43.42	43.28	44.87	43.57
15	69.00	67.18	68.26	68.34	67.34	68.27
16	50.00	54.83	53.22	53.00	55.59	53.38
17	66.33	67.16	67.82	67.88	67.73	68.10
18	66.00	63.84	64.81	64.92	63.82	64.85
19	59.33	54.64	55.88	55.91	53.66	55.56
20	70.33	64.94	66.42	66.48	63.85	66.00
21	65.67	64.70	65.34	65.49	64.67	65.50
22	55.33	62.56	56.72	56.09	61.89	55.43
23	70.33	68.16	69.15	69.26	68.17	69.26
24	81.33	80.07	80.87	81.06	80.02	81.06

25	62.33	71.82	70.09	69.96	73.18	70.51

Here in the above table every column represents the overall average marks a_i :
(a) using the raw average marks for 25 candidates
(b) using the Broyden's method of section 2.1
(c) using Broyden's method with a single fictitious candidate scoring 40 marks (c = 40) on each paper of section 2.2
(d) using Broyden's method with a modified loss (c =1600) of section 2.3
(e) using Biggins' et al. (1986) alternative method of section 2.4
(f) using Biggins' et al. (1986) modified method (taking c = 1.0) of section 2.5

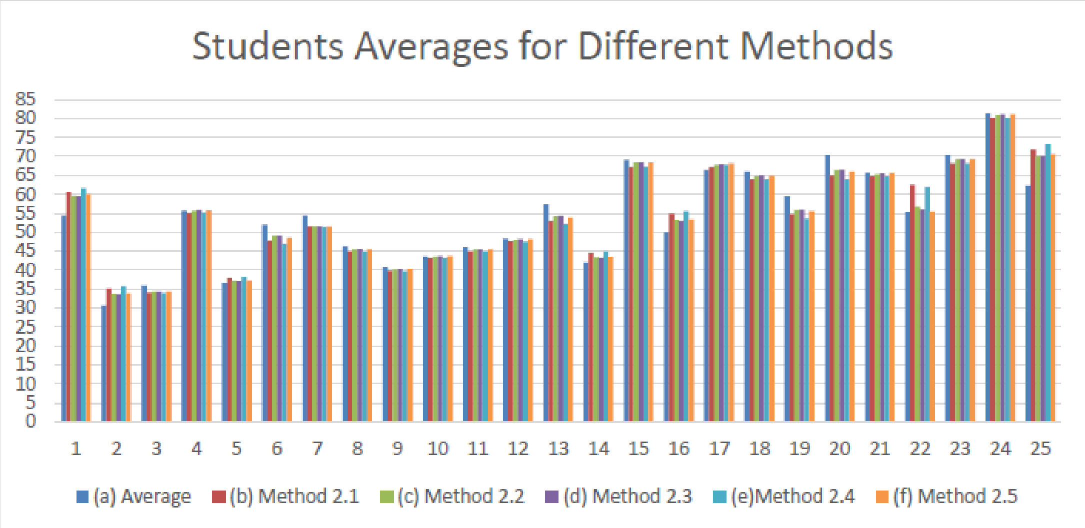
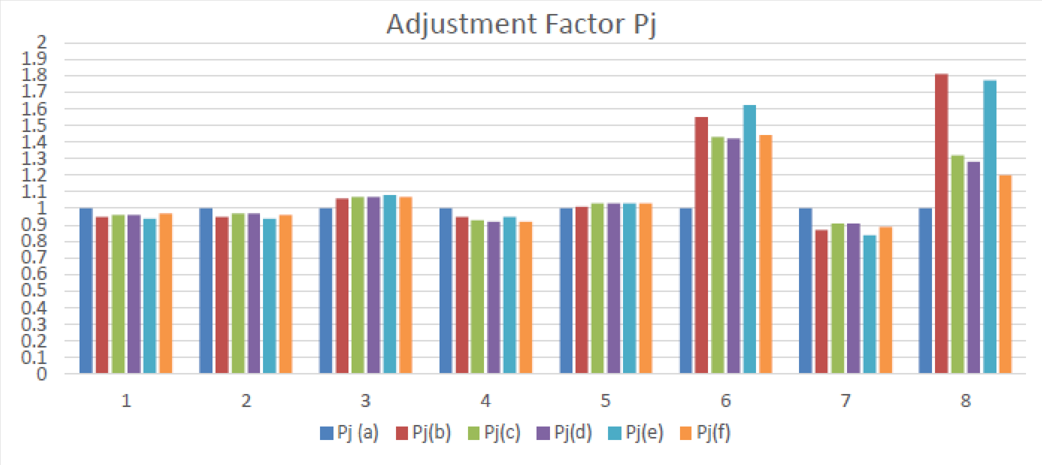


Table:3 The adjustment factors, p_j, for papers from 1 to paper 8 and the columns (a) to (f) corresponds to different methods.

Papers	(a)	(b)	(c)	(d)	(e)	(f)
1	1.0	0.95	0.96	0.96	0.94	0.97
2	1.0	0.95	0.97	0.97	0.94	0.96
3	1.0	1.06	1.07	1.07	1.08	1.07
4	1.0	0.95	0.93	0.92	0.95	0.92
5	1.0	1.01	1.03	1.03	1.03	1.03
6	1.0	1.55	1.43	1.42	1.62	1.44
7	1.0	0.87	0.91	0.91	0.84	0.89
8	1.0	1.81	1.32	1.28	1.77	1.20

Here in the above table every column represents the adjustment factor p_j :
(a) using the raw average marks for 25 candidates
(b) using Broyden's method of section 2.1
(c) using Broyden's method with a single fictitious candidate scoring 40 marks (c = 40) on each paper of section 2.2
(d) using Broyden's method with a modified loss (c =1600) of section 2.3
(e) using Biggins' et al. (1986) alternative method of section 2.4
(f) using Biggins' et al. (1986) modified method (taking c = 1.0) of section 2.5



IV. Discussion and Conclusion

A comparative analysis shows that paper-6 and Paper-8 may be difficult, or the examiner may be tough enough or the exam was from out of contents or were less popular because the students taking these papers were in fact having above average marks in other papers except these two. For example, the student-22, scored 72% (paper average: 57%) in the compulsory paper-1, 59% (paper average: 48%) in paper-4, and was the only who took paper-8 but scored on 35%. It shows that the student was excellent but due to taking an unpopular paper-8, he got only average 55%. Hence, he may be given some compensation or some extra benefits in average marks.

Similarly, the student-25, scored 73% (paper average: 57%) in the compulsory paper-1, 63% (paper average: 54%) in paper-3, and scored 51% in paper-6 (paper average: 32%). It shows that the student was also an excellent but due to taking an unpopular paper-6, he got only average 62%. Hence, he may be given some compensation or some extra benefits in average marks.

The paper-7 was although less popular but probably the paper was much easier or the examiner might be generous, because all those students, who took this paper-7 got above average marks. Hence, those student's average marks may be deducted who took paper-7.

We also noted that higher the number of compulsory papers, the differences in the averages using different method may be very small or ignorable. Whereas, if most of the papers/subjects are optional then the scaling process may produce large differences in the averages of the students. We reached to the conclusion that scaling methods give a fair indication of the abilities of the students/candidates.

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