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Stability and Bifurcation Analysis for the Dynamical Model of a New Three-dimensional Chaotic System

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Abstract
This paper analyzes the dynamic behaviors including stability and Hopf bifurcations of a class of three dimensional nonlinear chaotic systems with analytical and numerical methods. All the equilibriums and stability of the system are studied in detail for different values of system parameters. It is presented that there may exist three equilibriums for this system. The stability conditions of these equilibriums are obtained with the Routh–Hurwitz criterion. Using the Hopf bifurcation theorem and the first Lyapunov coefficient, the system condition and type of Hopf bifurcation for this system is obtained analytically. It is demonstrated that there may exist subcritical Hopf bifurcations under certain system parameters for this system. Using the Runge-Kutta method, numerical simulations including phase portraits and time history curves are also given, which verify the analytical results.
Keywords: *Chaotic system , Stability, Hopf Bifurcation*

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I. INTRODUCTION

Chaos is a common form of motion in nature. In 1963, the famous meteorologist Lorenz discovered the chaotic attractor of a three-dimensional autonomous system for the first time in a numerical experiment, and proposed the Lorenz chaotic system [1]. In 1999, Chen discovered a chaotic system similar to the Lorenz chaotic system but with a different topological structure, called the Chen system [2]. In 2004, Liu et al. proposed a new three-dimensional autonomous chaotic system called Liu system with nonlinear squared terms [3]. In 2004, Tigan et al. deleted the linear term of the Lorenz chaotic system and proposed a new chaotic system, namely T chaotic system [4].Wang et al. made a series of researches and analyses on the T chaotic system, for example, a simple dynamic analysis of the new three-dimensional chaotic system is carried out by using phase diagram, bifurcation and Lyapunov exponent [5]. In 2007, Wang proposed a new three-dimensional quadratic continuous autonomous chaotic system with only one system parameter [6]. At the same time, many scholars at home and abroad [7-12] successively discovered and proposed new chaotic systems and carried out research on chaotic control, synchronization and other related topics on this basis. This paper takes a new type of three-dimensional autonomous chaotic system as the research object. The study of theoretical derivation, numerical simulation and Lyapunov exponent reveals the rich dynamic characteristics of the system and provides an important basis for practical applications.

II. MODEL AND ANALYSIS OF NEW THREE-DIMENSIONAL CHAOTIC SYSTEM

A.*Dynamic Model of New Dynamic Analysis of the System*

Consider a new three-dimensional chaotic system [13], With the following dynamic equation:

$$\begin{cases} x_1 = a(x_2 - x_1) \\ x_2 = abx_1 - ax_1x_3 \\ x_3 = x_1x_2 - (a+c)x_3 \end{cases} \qquad (1)$$

Where $X^T = (x_1, x_2, x_3) \in \mathbf{R}^3$ are state variables, and a, b and c are system parameters.

The system can be expressed as $X = AX + f(X)$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} -a & a & 0 \\ ab & 0 & 0 \\ 0 & 0 & -(a+c) \end{pmatrix}; \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad f(X) = \begin{pmatrix} 0 \\ -ax_1x_3 \\ x_1x_2 \end{pmatrix}$$

It can be seen that in this system $a_{11}a_{22} = 0$. Since the Lorenz system is $a_{11}a_{22} > 0$ and the Liu system is $a_{11}a_{22} < 0$, and this system is not topologically equivalent to the above-mentioned classical system, the three-dimensional autonomous system is a new critical system among the classical chaotic systems.

B.*Dynamic Analysis of the System*

The Equilibrium Points of the System and Stability Analysis

Letting the value of the right end of the equation (1) equal to 0:

$$\begin{cases} x_1 = x_2 \\ a x_1 (b - x_3) = 0 \\ x_1 x_2 - (a + c) x_3 = 0 \end{cases} \qquad (2)$$

we can get the equilibrium points of the system as follows:

$$O(0,0,0,); \quad P_+(\sqrt{a(b+c)}, \sqrt{a(b+c)}, b); \quad P_-(-\sqrt{a(b+c)}, -\sqrt{a(b+c)}, b)$$

The Jacobian matrix evaluated at the point $(x, y, z) = (x_{eq}, y_{eq}, z_{eq})$ is:

$$M = \begin{pmatrix} -a & a & 0 \\ ab - az_{eq} & 0 & -ax_{eq} \\ y_{eq} & x_{eq} & -(a+c) \end{pmatrix}$$

the characteristic polynomial is as follows:

$$f(\lambda) = \lambda^3 + (2a + c)\lambda^2 + (ax_{eq}{}^2 + a^2 z_{eq} + a^2 + ac - a^2 b)\lambda - a^3 b + a^3 z_{eq} - a^2 bc + a^2 c z_{eq} + x_{eq}{}^2 a^2 + a^2 x_{eq} y_{eq}$$

The system's characteristic equation is at point $O(0,0,0)$ is:

$$\lambda^3 + (2a + c)\lambda^2 + \left(a^2 + ac - a^2 b\right)\lambda - a^3 b - a^2 bc = 0$$

when $1 + 4b > 0$, the eigenvalues are as follows:

$$\lambda_1 = -a - c, \quad \lambda_2 = a(-\frac{1}{2} + \frac{\sqrt{1+4b}}{2}), \quad \lambda_3 = a(-\frac{1}{2} - \frac{\sqrt{1+4b}}{2})$$

Next, we consider the equilibrium point P_+, P_-. The difference between them is only a negative sign of the variable x_1, x_2. And if we turn x_1, x_2 into $-x_1, -x_2$ the system equation is kept unchanged in the system (1), so the stability of the system is the same in P_+, P_-. Next, we only need to consider the stability at point P_+. The Jacobian matrix of the system at point P_+ is:

$$\begin{pmatrix} -a & a & 0 \\ 0 & 0 & -a\sqrt{a(b+c)} \\ \sqrt{a(b+c)} & \sqrt{a(b+c)} & -a-c \end{pmatrix} \qquad (3)$$

The characteristic equation is:

$$f(\lambda) = \lambda^3 + (2a + c)\lambda^2 + \left(a^2 c + a^2 b + a^2 + ac\right)\lambda + 2a^3(b + c) \qquad (4)$$

From Routh–Hurwitz criterion Theorem [14], We know that the equilibrium point P_+ is asymptotically stable, if and only if the following conditions are satisfied:

$$\begin{cases} a^3 b + 2a^3 c - a^2 bc > o \\ (2a+c)\left(a^2 c + a^2 + ac + a^2 b\right) - (2a^3 b + 2a^3 c) > 0 \\ 2a + c > 0 \end{cases} \qquad (5)$$

III. HOPF BIFURCATION ANALYSIS

Hopf bifurcation is a dynamic bifurcation, which is a limit cycle caused by the instability of the equilibrium point, but where the stability changes, its eigenvalue must be a pure virtual root. Because there is no characteristic root similar to $\lambda_{1、2} = \pm iw(w > 0)$ at the equilibrium point $O(0,0,0)$, So Hopf bifurcation will not occur at the point $O(0,0,0)$. Therefore, it is only possible to produce Hopf bifurcation at the point P_+, P_-, and from symmetry, we will only analyze the point P_+.According to the Hopf bifurcation theory [15], when the system (1) contains a pair of complex eigenvalues that satisfy the following conditions, the system will generate Hopf bifurcation:

$$\begin{cases} \mathrm{Re}(\lambda)|_{b=b_0} = 0 \\ \mathrm{Im}(\lambda)|_{b=b_0} \neq 0 \\ \dfrac{d}{dx}\mathrm{Re}(\lambda)|_{b=b_0} \neq 0 \end{cases} \quad (6)$$

Where b_0 is the critical value of b when the system generates bifurcation.

If equation (4) has pure imaginary roots $\lambda_{1、2} = \pm iw(w > 0)$,we can obtain:

$$(iw)^3 + (2a + c)(iw)^2 + (a^2b + a^2c + a^2 + ac)(iw) + 2a^3b + 2a^3c = 0$$

Separating the real and imaginary parts of the above formula, we can obtain:

$$\begin{cases} w^2 = a^2b + a^2c + a^2 + ac \\ w^2 = \dfrac{2a^3b + 2a^3c}{2a + c} \end{cases}$$

For the existence of positive a、 b、 c and w , the parameters must meet the following conditions:

$$\begin{cases} 2a^3b + 2a^3c > 0 \\ 2a + c > 0 \\ a^2b + a^2c + a^2 + ac > 0 \\ \dfrac{2a^3b + 2a^3c}{2a + c} = a^2b + a^2c + a^2 + ac \end{cases} \quad (7)$$

We get the critical value:

$$b_0 = -\frac{2a^2 + ac^2 + 3ac + c^2}{ac} \quad (8)$$

The eigenvalues corresponding to the equilibrium points P_+ are:

$$\begin{aligned} &\lambda_1 = -(2a + c) \\ &\lambda_{2、3} = \pm i\sqrt{a^2b + a^2c + a^2 + ac} = \pm iw_h \end{aligned} \quad (9)$$

And

$$\lambda'(b) = -\frac{a^2\lambda + 2a^3}{3\lambda^2 + 2(2a + c)\lambda + (a^2c + a^2b + a^2 + ac)}$$

So we can obtain:

$$\mathrm{Re}\,\lambda'(b_0) = \frac{2(a^2w_h + 2a^3)}{4w_h^2 + (4a + 2c)^2} \neq 0 \quad (10)$$

From the equation (7),we know that the stability changes on both sides b_0 , so the Hopf bifurcation would happen.In summary, when $b > b_0$,the equilibrium point is stable. While $b < b_0$, the equilibrium point becomes unstable. In the following, we use the first Lyapunov coefficient[16] to further discuss the supercriticality or subercriticality of Hopf bifurcation. Suppose *Cn* is a linear space defined in the complex field *C* with inner product, for any vector $x=(x_1,x_2,\cdots,x_n)^T$, $y=(y_1,y_2,\cdots,y_n)^T$,where $x_i,y_i\in C$, , $(i=1,2\cdots n)$, $\langle x_i,y_i\rangle=\sum_{i=1}^{n}\overline{x_i}y_i$ we introduce norm $\|x\|=\sqrt{\langle x,x\rangle}$,so that *Cn* is a Hilbert space. Consider the following nonlinear system:

$$\dot{x}=Ax+F(x),x\in R^n$$

Where $F(x)=o\left(\|x^2\|\right)$ is a smooth function, and it can be expanded into:

$$F(x)=\frac{1}{2}B(x,x)+\frac{1}{6}C(x,x,x)+o\left(\|x^4\|\right)$$

in which $B(x,x)$ and $C(x,x,x)$ are bilinear and trilinea functions, respectively. In equation (1), If the matrix M has a pair of pure imaginary eigenvalues $\lambda_{2、3}=\pm iw(w>0)$, Let $q\in \mathbf{C}^n$ be a complex eigenvector corresponding to the eigenvalue λ_2 ,then we can get $Mq=iwq$, $M\overline{q}=-iw\overline{q}$,Meanwhile,we introduce the adjoint vector $p\in C^n$ which satisfies $M^Tp=-iwp$, $M^Tp=iw\overline{p}$ and $\langle p,q\rangle=1$.

Introducing the transformation as follows:

$$\begin{cases} X=x-x_{eq} \\ Y=y-y_{eq} \\ Z=z-z_{eq} \end{cases}$$

then the equilibrium point (x_{eq},y_{eq},z_{eq}) transform to the origin $O(0,0,0,)$.

According to the first Lyapunov coefficient theorem at the equilibrium point of the system [17-18]:

$$l_1(0)=\frac{1}{2w}\mathrm{Re}\left(\left\langle p,C\left(q,q,\overline{q}\right)\right\rangle-2\left\langle p,B\left(q,M^{-1}B\left(q,\overline{q}\right)\right)\right\rangle+\left\langle p,B\left(\overline{q},(2iwE-M)^{-1}B(q,q)\right)\right\rangle\right)$$

Choosing the system parameters $a=1,c=-\frac{1}{2}$,then we can obtain that $b_0=2$ and $P_+\left(\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}},2\right)$,as well as the Jacobian matrix of the system (1):

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & -\sqrt{\frac{3}{2}} \\ -\sqrt{\frac{3}{2}} & -\sqrt{\frac{3}{2}} & -\frac{1}{2} \end{pmatrix}$$

Calculating the corresponding vectors *p*, *q* of matrix *J* satisfy $Mq=iwq$, $M^Tp=-iwp$, and $\langle p,q\rangle=1$,we can obtain:

$$q=\left(\frac{\sqrt{3}i}{2(1+\sqrt{2}i)},\frac{1}{2}\sqrt{3}i,1\right)^T$$

$$p = \left(\frac{\sqrt{6}\left(\sqrt{2}i-1\right)}{2\left(5\sqrt{2}i+1\right)}, -\frac{\sqrt{3}i\left(\sqrt{2}i-1\right)^2\left(\sqrt{2}i+4\right)}{6(5\sqrt{2}i+1)}, -\frac{(\sqrt{2}i-1)^2}{5\sqrt{2}i+1} \right)^T \quad \bar{q} = \left(\frac{\sqrt{3}i}{2(\sqrt{2}i-1)}, \ -\frac{1}{2}\sqrt{3}i, 1 \right)^T$$

Where $\bar{q}$ is the conjugate vector of q .

In addition, the bilinear and trilinear functions for the system:

$$B\left(x, x'\right) = \left(0, -axz', xy'\right)^T$$

$$C\left(x, x', x''\right) = \left(0, 0, 0\right)^T$$

Then:

$$B\left(q, q\right) = \left(0, -\frac{\sqrt{3}i}{2(1+\sqrt{2}i)}, \frac{\sqrt{3}i}{4(1+\sqrt{2}i)}\right)$$

$$B\left(q, \bar{q}\right) = \left(0, -\frac{\sqrt{3}i}{2(1+\sqrt{2}i)}, -\frac{\sqrt{3}i}{4(1+\sqrt{2}i)}\right)$$

The inverse of matrix M is:

$$M^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{6} & \frac{\sqrt{6}}{6} \\ \frac{1}{2} & -\frac{1}{6} & \frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{6}}{3} & 0 \end{pmatrix} \qquad (10)$$

Considering $s = M^{-1}B\left(q, \bar{q}\right)$, one can obtain that:

$$s = \left(\frac{\left(2\sqrt{3}-3\sqrt{2}\right)i}{24\left(\sqrt{2}i+1\right)}, \frac{\left(2\sqrt{3}-3\sqrt{2}\right)i}{24\left(\sqrt{2}i+1\right)}, \frac{\sqrt{2}i}{2\left(1+\sqrt{2}i\right)} \right)$$

And

$$B\left(q, s\right) = \left(0, \frac{\sqrt{6}}{4\left(\sqrt{2}i+1\right)^2}, -\frac{6-3\sqrt{6}}{48\left(\sqrt{2}i+1\right)^2}\right)^T$$

Therefore, it can be obtained that:

$$\left\langle p, B\left(q, s\right)\right\rangle = \frac{-6+\sqrt{6}-8\sqrt{2}i}{16\left(-1+5\sqrt{2}i\right)}$$

Due to $s' = \left(2iwE - M\right)^{-1}B\left(q, q\right)$, it can be deduced that:

$$s' = \left(\frac{14\sqrt{5}\left(48i-19\sqrt{6}\right)}{45\left(5\sqrt{6}i+8\right)\left(\sqrt{6}i+2\right)}, -\frac{\sqrt{5}\left(253\sqrt{6}+504i\right)}{45\left(5\sqrt{6}i+8\right)\left(\sqrt{6}i+2\right)}, \frac{8\left(14\sqrt{6}i-27\right)}{9\left(5\sqrt{6}i+8\right)\left(\sqrt{6}i+2\right)} \right)^T$$

and

$$B\left(\bar{q}, s'\right) = \left(0, \frac{\sqrt{6}i\left(2\sqrt{6}i+3i+\sqrt{3}-3\sqrt{2}\right)}{36\left(4\sqrt{2}i+3\right)}, \frac{\left(2\sqrt{6}i+\sqrt{3}\right)\left(8\sqrt{6}i+2\sqrt{3}+3\sqrt{2}\right)}{144\left(4\sqrt{2}i+3\right)}\right)^T$$

Therefore

$$\left\langle p, B\left(\bar{q}, s^{'}\right)\right\rangle = \frac{\left(\sqrt{2}i+1\right)^{2}\left(72\sqrt{2}i+40\sqrt{3}i+19\sqrt{6}-126\right)}{144\left(5\sqrt{2}i-1\right)\left(4\sqrt{2}i+3\right)}$$

Consequently, it is obtained that:

$$l_1(0) = \frac{1}{2w}\mathrm{Re}\left(\left\langle p, C\left(q,q,\bar{q}\right)\right\rangle - 2\left\langle p, B\left(q, M^{-1}B\left(q,\bar{q}\right)\right)\right\rangle + \left\langle p, B\left(\bar{q},(2iwE-M)^{-1}B\left(q,q\right)\right)\right\rangle\right)$$

$$= \frac{1}{2\sqrt{2}}\left(\frac{599\sqrt{6}}{33456}+\frac{3025}{10728}\right) \approx 0.07944007738 > 0$$

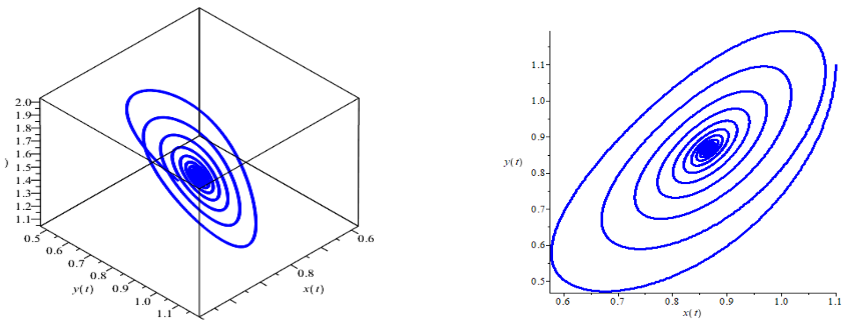
Therefore, Hopf bifurcation obtained under this set of parameters is subcritical.

IV. NUMERICAL SIMULATIONS

We numerically simulate the system (1) by using Runge- Kutta numerical calculation method. For the system,choosing the following parameters: $a=1, c=-\frac{1}{2}$,thus, when the bifurcation occurs, the critical value is $b_0=2$.

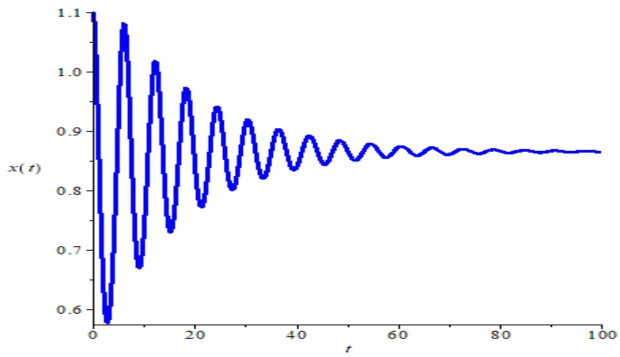
1. when $b=\frac{3}{2}<b_0$, the phase portrait of the projection in $x-y-z$ and $x-y$ plane are shown as in Figure 1 (a),(b) respectively. In addition, Trajectories of $x(t)$ are shown as in Figure 2 respectively,form which we can see that the equilibrium is stable.

2.



(a) the phase portrait of $x(t)$, $y(t)$,$z(t)$ **(b)**the phase portrait of $x(t)$, $y(t)$

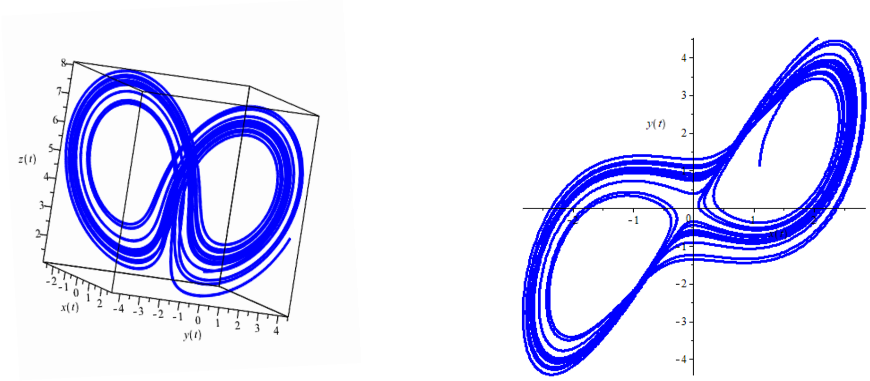
Figure1.the phase portrait for $a=1, c=-\frac{1}{2}, b=\frac{3}{2}$



the trajectories of $x(t)$

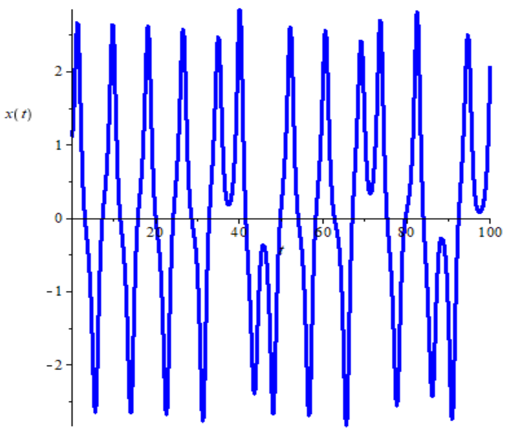
Figure **2**. Trajectories for , $a=1, c=-\frac{1}{2}\ b=\frac{3}{2}$

3. When $b = 5 > b_0$,we can get the graph shown as in Figure 3 and Figure 4, respectively.



(a)the phase portrait of $x(t)$, $y(t)$,$z(t)$ **(b)**the phase portrait of $x(t)$, $y(t)$

Figure **3**. Trajectories for $a = 1, c = -\frac{1}{2}, b = 5$



the trajectories of $x(t)$

Figure **4**. Trajectories for $a = 1, c = -\frac{1}{2}, b = 5$

It can be seen from the previous analysis that when $b < b_0$,the equilibrium point is asymptotically stable, and when $b > b_0$,the system will generate Hopf bifurcation, and the numerical results are consistent with the theoretical analysis results.

V. CONCLUSION

This paper studies a new three-dimensional chaotic system and analyzes its equilibrium point and stability. Hopf bifurcation theorem and the first Lyapunov coefficient theorem are used to investigate the conditions and types of bifurcation in the system.There may be a subcritical Hopf branch under certain system parameters of the system. Numerical simulations verify the analytical results. For the new autonomous chaotic system, there are still many problems worthy of further study and discovery, such as some more complex dynamic characteristics, the circuit realization of the system, the control and synchronization of chaos and its application in secure communication, etc. The results obtained here can provide some guidance for the analysis and design of secure communication systems. This is of great significance in real life, and this is also a subject that the author will further study in depth.

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