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On The Non-Homogeneous Quintic Equation With Five Unknowns $3(x+y)(x^3-y^3)=7(z^2-w^2)p^3$

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Abstract: *The quintic non-homogeneous equation with five unknowns represented by the Diophantine equation* $3(x+y)(x^3-y^3)=7(z^2-w^2)p^3$ *is analyzed for its patterns of non-zero distinct integral solutions.*

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I. INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems [1-4]. Particularly, in [5-8] quintic equations with three unknowns are studied for their integral solutions. In [9,10] quintic equations with four unknowns for their non-zero integer solutions are analyzed. [11-15] analyze quintic equations with five unknowns for their non-zero integer solutions. This communication concerns with yet another interesting non-homogeneous quintic equation with five unknowns given by $3(x+y)(x^3-y^3)=7(z^2-w^2)p^3$ for finding its infinitely many non-zero distinct integer solutions.

II. METHOD OF ANALYSIS

The non-homogeneous quintic equation with five unknowns to be solved for its distinct non-zero integral solutions is

$$3(x+y)(x^3-y^3)=7(z^2-w^2)p^3 \tag{1}$$

METHOD 1:

Introduction of the linear transformations

$$x=u+v, y=u-v, z=3u+v, w=3u-v \tag{2}$$

in (1) leads to

$$v^2+3u^2=7P^3 \tag{3}$$

Different methods of obtaining the patterns of integer solutions to (1) are illustrated below:

PATTERN: 1

Let

$$P=a^2+3b^2 \tag{4}$$

where a and b are non-zero integers.

Write 7 as

$$7=(2+i\sqrt{3})(2-i\sqrt{3}) \tag{5}$$

Using (4), (5) in (3) and applying the method of factorization, define

$$(v+i\sqrt{3}u)=(2+i\sqrt{3})(a+i\sqrt{3}b)^3 \tag{6}$$

from which we have

$$\left.\begin{aligned} v &= 2a^3 - 18ab^2 - 9a^2b + 9b^3 \\ u &= a^3 - 9ab^2 + 6a^2b - 6b^3 \end{aligned}\right\} \tag{7}$$

Using (7) and (2), the values of x, y, z and w are given by

$$\left.\begin{aligned} x(a,b) &= 3a^3 + 3b^3 - 27ab^2 - 3a^2b \\ y(a,b) &= -a^3 - 15b^3 + 9ab^2 + 15a^2b \\ z(a,b) &= 5a^3 - 9b^3 - 45ab^2 + 9a^2b \\ w(a,b) &= a^3 - 27b^3 - 9ab^2 + 27a^2b \end{aligned}\right\} \tag{8}$$

Thus (4) and (8) represent the non-zero integer solutions to (1).

PATTERN: 2

Write 7 as

$$7 = \frac{(5 + i\sqrt{3})(5 - i\sqrt{3})}{4} \tag{9}$$

Using (4), (9) in (3) and applying the method of factorization, define

$$(v + i\sqrt{3}u) = \frac{(5 + i\sqrt{3})}{2}(a + i\sqrt{3}b)^3 \tag{10}$$

from which we have

$$\left.\begin{aligned} v &= \frac{1}{2}\left[5a^3 + 9b^3 - 45ab^2 - 9a^2b\right] \\ u &= \frac{1}{2}\left[a^3 - 15b^3 - 9ab^2 + 15a^2b\right] \end{aligned}\right\} \tag{11}$$

Using (11) and (2), the values of x, y, z and w are given by

$$\left.\begin{aligned} x(a,b) &= 3a^3 - 3b^3 - 27ab^2 + 3a^2b \\ y(a,b) &= -2a^3 - 12b^3 + 18ab^2 + 12a^2b \\ z(a,b) &= 4a^3 - 18b^3 - 36ab^2 + 18a^2b \\ w(a,b) &= -a^3 - 27b^3 + 9ab^2 + 27a^2b \end{aligned}\right\} \tag{12}$$

Thus (4) and (12) represent the non-zero integer solutions to (1).

PATTERN:3

Write (3) as

$$v^2 + 3u^2 = 7p^3 * 1 \tag{13}$$

Write 1 as

$$1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \tag{14}$$

Using (4), (5),(14) in (13) and applying the method of factorization, define

$$(v + i\sqrt{3}u) = \frac{1}{2}(1 + i\sqrt{3})(2 + i\sqrt{3})(a + i\sqrt{3}b)^3 \tag{15}$$

from which we have

$$\left.\begin{aligned} v &= \frac{1}{2}\left[-a^3 + 27b^3 + 9ab^2 - 27a^2b\right] \\ u &= \frac{1}{2}\left[3a^3 + 3b^3 - 27ab^2 - 3a^2b\right] \end{aligned}\right\} \tag{16}$$

Using (16) in (2), the values of x, y, z and w are given by

$$\left.\begin{aligned} x(a,b) &= a^3 + 15b^3 - 9ab^2 - 15a^2b \\ y(a,b) &= 2a^3 - 12b^3 - 18ab^2 + 12a^2b \\ z(a,b) &= 4a^3 + 18b^3 - 36ab^2 - 18a^2b \\ w(a,b) &= 5a^3 - 9b^3 - 45ab^2 + 9a^2b \end{aligned}\right\} \tag{17}$$

Thus (4) and (17) represents the non-zero integer solutions to (1).

PATTERN: 4

Write 1 as

$$1=\frac{\left(1+i4\sqrt{3}\right)\left(1-i4\sqrt{3}\right)}{49} \tag{18}$$

Using (4), (5) and (18) in (13) and applying the method of factorization, define

$$\left(v+i\sqrt{3}u\right)=\frac{1}{7}\left(2+i\sqrt{3}\right)\left(1+i4\sqrt{3}\right)\left(a+i\sqrt{3}b\right)^{3} \tag{19}$$

from which we have

$$\left.\begin{aligned} v&=\frac{1}{7}\left[-10a^{3}+81b^{3}+90ab^{2}-81a^{2}b\right]\\ u&=\frac{1}{7}\left[9a^{3}+30b^{3}-81ab^{2}-30a^{2}b\right]\end{aligned}\right\} \tag{20}$$

Since our interest is on finding integer solutions, replacing a by 7A, b by 7B in (4) and (20) & using (2), the corresponding integer solutions to (1) are given by

$$\left.\begin{aligned} x(A,B)&=7^{2}\left(-A^{3}+111B^{3}+9AB^{2}-111A^{2}B\right)\\ y(A,B)&=7^{2}\left(19A^{3}-51B^{3}-171AB^{2}+51A^{2}B\right)\\ z(A,B)&=7^{2}\left(17A^{3}+171B^{3}-153AB^{2}-171A^{2}B\right)\\ w(A,B)&=7^{2}\left(37A^{3}+9B^{3}-333AB^{2}-9A^{2}B\right)\\ p(A,B)&=7^{2}\left(A^{2}+3B^{2}\right)\end{aligned}\right\} \tag{21}$$

Thus (21) represents the non-zero integer solutions to (1).

PATTERN: 5

Write 1 as

$$1=\frac{\left(1+i\sqrt{3}\right)\left(1-i\sqrt{3}\right)}{4} \tag{22}$$

Using (4), (9) and (22) in (13) and applying the method of factorization, define

$$\left(v+i\sqrt{3}u\right)=\frac{1}{4}\left(5+i\sqrt{3}\right)\left(1+i\sqrt{3}\right)\left(a+i\sqrt{3}b\right)^{3} \tag{23}$$

from which we have

$$\left.\begin{aligned} v&=\frac{1}{2}\left[a^{3}+27b^{3}-9ab^{2}-27a^{2}b\right]\\ u&=\frac{1}{2}\left[3a^{3}-3b^{3}-27ab^{2}+3a^{2}b\right]\end{aligned}\right\} \tag{24}$$

Using (24) in (2), the values of x,y,z and w are given by

$$\left.\begin{aligned} x(a,b)&=2a^{3}+12b^{3}-18ab^{2}-12a^{2}b\\ y(a,b)&=a^{3}-15b^{3}-9ab^{2}+15a^{2}b\\ z(a,b)&=5a^{3}+9b^{3}-45ab^{2}-9a^{2}b\\ w(a,b)&=4a^{3}-18b^{3}-36ab^{2}+18a^{2}b\end{aligned}\right\} \tag{25}$$

Thus (4) and (25) represents the non-zero integer solutions to (1).

METHOD 2:

Introduction of the linear transformations

$$x=u+v,\ y=u-v,\ z=u+3v,\ w=u-3v \tag{26}$$

in (1) leads to (3).

Following the same process from Pattern 1 to Pattern 5 and using the transformation (26), the sets of solutions to (1) are given below in Table 1:

Table 1: Solutions

Patterns	Solutions
1	$x(a,b)=3a^3+3b^3-27ab^2-3a^2b$ $y(a,b)=-a^3-15b^3+9ab^2+15a^2b$ $z(a,b)=7a^3+21b^3-63ab^2-21a^2b$ $w(a,b)=-5a^3-33b^3+45ab^2+33a^2b$ $P(a,b)=a^2+3b^2$
2	$x(a,b)=3a^3-3b^3-27ab^2+3a^2b$ $y(a,b)=-2a^3-12b^3+18ab^2+12a^2b$ $z(a,b)=8a^3+6b^3-72ab^2-6a^2b$ $w(a,b)=-7a^3-21b^3+63ab^2+21a^2b$ $P(a,b)=a^2+3b^2$
3	$x(a,b)=a^3+15b^3-9ab^2-15a^2b$ $y(a,b)=2a^3-12b^3-18ab^2+12a^2b$ $z(a,b)=42b^3-42a^2b$ $w(a,b)=3a^3-39b^3-27ab^2+39a^2b$ $P(a,b)=a^2+3b^2$
4	$x(A,B)=7^2(-A^3+111B^3+9AB^2-111A^2B)$ $y(A,B)=7^2(19A^3-51B^3-171AB^2+51A^2B)$ $z(A,B)=7^2(-21A^3+273B^3+189AB^2-273A^2B)$ $w(A,B)=7^2(39A^3-213B^3-351AB^2+213A^2B)$ $p(A,B)=7^2(A^2+3B^2)$
5	$x(a,b)=2a^3+12b^3-18ab^2-12a^2b$ $y(a,b)=a^3-15b^3-9ab^2+15a^2b$ $z(a,b)=3a^3+39b^3-27ab^2-39a^2b$ $w(a,b)=42a^2b-42b^3$ $P(a,b)=a^2+3b^2$

METHOD 3:

Introduction of the linear transformations

$$x=u+v, y=u-v, z=3uv+1, w=3uv-1 \qquad (27)$$

in (1) leads to (3).

Following the same process from Pattern 1 to Pattern 5 and using the transformation (27), the sets of solutions to (1) are given below in Table 2:

Table 2: Solutions

Patterns	Solutions
1	$x(a,b)=3a^3+3b^3-27ab^2-3a^2b$ $y(a,b)=-a^3-15b^3+9ab^2+15a^2b$ $z(a,b)=6f^2(a,b)+9f(a,b)g(a,b)-162g^2(a,b)+1$ $w(a,b)=6f^2(a,b)+9f(a,b)g(a,b)-162g^2(a,b)-1$ where $f(a,b)=a^3-9ab^2$ and $g(a,b)=a^2b-b^3$ $P(a,b)=a^2+3b^2$

2	$x(a,b)=3a^3-3b^3-27ab^2+3a^2b$ $y(a,b)=-2a^3-12b^3+18ab^2+12a^2b$ $z(a,b)=48[5f^2(a,b)+66f(a,b)g(a,b)-135g^2(a,b)]+1$ $w(a,b)=48[5f^2(a,b)+66f(a,b)g(a,b)-135g^2(a,b)]-1$ where $f(a,b)=a^3-9ab^2$ and $g(a,b)=a^2b-b^3$ $P(a,b)=a^2+3b^2$
3	$x(a,b)=a^3+15b^3-9ab^2-15a^2b$ $y(a,b)=2a^3-12b^3-18ab^2+12a^2b$ $z(a,b)=48[-3f^2(a,b)-78f(a,b)g(a,b)+81g^2(a,b)]+1$ $w(a,b)=48[-3f^2(a,b)-78f(a,b)g(a,b)+81g^2(a,b)]-1$ where $f(a,b)=a^3-9ab^2$ and $g(a,b)=a^2b-b^3$ $P(a,b)=a^2+3b^2$
4	$x(A,B)=7^2(-A^3+111B^3+9AB^2-111A^2B)$ $y(A,B)=7^2(19A^3-51B^3-171AB^2+51A^2B)$ $z(A,B)=147[-90f^2(A,B)-429f(A,B)g(A,B)+2430g^2(A,B)]+1$ $w(A,B)=147[-90f^2(A,B)-429f(A,B)g(A,B)+2430g^2(A,B)]-1$ where $f(A,B)=A^3-9AB^2$ and $g(A,B)=A^2B-B^3$ $p(A,B)=7^2(A^2+3B^2)$
5	$x(a,b)=2a^3+12b^3-18ab^2-12a^2b$ $y(a,b)=a^3-15b^3-9ab^2+15a^2b$ $z(a,b)=48[3f^2(a,b)-78f(a,b)g(a,b)-81g^2(a,b)]+1$ $w(a,b)=48[3f^2(a,b)-78f(a,b)g(a,b)-81g^2(a,b)]-1$ where $f(a,b)=a^3-9ab^2$ and $g(a,b)=a^2b-b^3$ $P(a,b)=a^2+3b^2$

III. CONCLUSION

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous quintic equations with five unknowns given by $3(x+y)(x^3-y^3)=7(z^2-w^2)p^3$. As the quintic equations are rich in variety, one may search for other forms of quintic equation with variables greater than or equal to five.

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