Research Inventy: International Journal of Engineering And Science
Vol.10, Issue 8 (August 2020), PP 13-18
Issn (e): 2278-4721, Issn (p):2319-6483, www.researchinventy.com

A Search On the Integer Solutions of Cubic Diophantine Equation with Four Unknowns $x^3 - y^3 = 4(w^3 - z^3) + 3(x - y)^3$

S.Vidhyalakshmi[1], T. Mahalakshmi[2] ,M.A.Gopalan [3*]
[1,2] Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.
[3] Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, Trichy-620 002, Tamil Nadu, India.
Corresponding Author: M.A.Gopalan

ABSTRACT: *The cubic diophantine equation with four unknowns given by* $x^3 - y^3 = 4(w^3 - z^3) + 3(x - y)^3$ *is analyzed for its non-zero distinct integral solutions. Using different choices , integer solutions for the equation under consideration are obtained.*

KEYWORDS: *Cubic equation with four unknowns, Integral solutions.*

Date of Submission: 24-07-2020 Date of Acceptance: 05-08-2020

I. INTRODUCTION

The cubic diophantine equations are rich in variety and offer an unlimited field for research [1,2]. For an extensive review of various problems, one may refer [3-24]. This paper concerns with another interesting cubic diophantine equation with four unknowns given by $x^3 - y^3 = 4(w^3 - z^3) + 3(x - y)^3$ is analysed for determining its infinitely many non-zero integral solutions.

II. METHOD OF ANALYSIS

The cubic diophantine equation to be solved for its non-zero distinct integral solutions is given by

$$x^3 - y^3 = 4(w^3 - z^3) + 3(x - y)^3 \tag{1}$$

Introducing the linear transformations

$$x = u + v, \quad y = u - v, \quad w = p + v, \; z = p - v \quad u \neq v \neq p \tag{2}$$

in (1), it changes to

$$u^2 = 5v^2 + 4p^2 \tag{3}$$

We present different methods of solving (3) to get different sets of integer solutions to (1).

METHOD 1:

We can write (3) in the form of ratio as

$$\frac{u + 2p}{v} = \frac{5v}{u - 2p} = \frac{\alpha}{\beta}, \beta \neq 0$$

The above equation is equivalent to the double equations

$$\beta u - \alpha v + 2\beta p = 0 \quad \text{and}$$

$$\alpha u - 5\beta v - 2\alpha p = 0$$

Applying the method of cross multiplication, we get

$$u = 2\alpha^2 + 10\beta^2 \;, \; p = \alpha^2 - 5\beta^2 \;, \; v = 4\alpha\beta \tag{4}$$

Using (4) in (2), we have

$$\left.\begin{aligned} x &= 2\alpha^2 + 10\beta^2 + 4\alpha\beta \\ y &= 2\alpha^2 + 10\beta^2 - 4\alpha\beta \\ w &= \alpha^2 - 5\beta^2 + 4\alpha\beta \\ z &= \alpha^2 - 5\beta^2 - 4\alpha\beta \end{aligned}\right\} \quad (5)$$

Thus, (5) gives a set of integer solutions for (1).
Note: One may write (3) also in the form of ratio as

$$\frac{u+2p}{5v} = \frac{v}{u-2p} = \frac{\alpha}{\beta}, \beta \neq 0$$

In this case, the corresponding integer solutions to (1) are given by

$$x = 10\alpha^2 + 2\beta^2 + 4\alpha\beta$$

$$y = 10\alpha^2 + 2\beta^2 - 4\alpha\beta$$

$$w = 5\alpha^2 - \beta^2 + 4\alpha\beta$$

$$z = 5\alpha^2 - \beta^2 - 4\alpha\beta$$

METHOD 2:
We write (3) as a system of double equations as shown in Table 1 below:

Table 1: System of Double Equations

System	I	II
$u+2p$	v^2	$5v^2$
$u-2p$	5	1

After some algebra, the solutions to (1) through solving each of the systems in Table 1 are exhibited as follows:
Solutions through system I:

$$x = 2k^2 + 4k + 4, y = 2k^2 + 2, z = 2k^2 + 4k, w = 2k^2 - 2$$

Solutions through system II:

$$x = 10k^2 + 12k + 4, y = 10k^2 + 8k + 2, z = 5k^2 + 3k, w = 5k^2 + 7k + 2$$

METHOD 3:
Let

$$p = a^2 - 5b^2 \quad (6)$$

where a and b are non-zero integers.
Write 4 as

$$4 = \left(3+\sqrt{5}\right)\left(3-\sqrt{5}\right) \quad (7)$$

Using (6), (7) in (3) and applying the method of factorization, define

$$\left(u+\sqrt{5}v\right) = \left(3+\sqrt{5}\right)\left(a+\sqrt{5}b\right)^2$$

from which we have

$$\left.\begin{aligned} u &= 3a^2 + 10ab + 15b^2 \\ v &= a^2 + 6ab + 5b^2 \end{aligned}\right\} \quad (8)$$

Using (6) and (8) in (2) the values of x, y, w and z satisfying (1) are given by

$$x = 4a^2 + 16ab + 20b^2$$

$$y = 2a^2 + 4ab + 10b^2$$

$$w = 2a^2 + 6ab$$

$$z = -6ab - 10b^2$$

Note: In addition to (7), 4 may also be written as below:

$$4 = \left(7+3\sqrt{5}\right)\left(7-3\sqrt{5}\right) \quad (9)$$

$$4=\frac{\left(23+3\sqrt{5}\right)\left(23-3\sqrt{5}\right)}{121} \tag{10}$$

Using (6), (9) and (2), the values of x, y, z and w satisfying (1) are given by

$$x = 10a^2 + 44ab + 50b^2$$
$$y = 4a^2 + 16ab + 20b^2$$
$$z = 4a^2 + 14ab + 10b^2$$
$$w = -2a^2 - 14ab - 20b^2$$

Using (6),(10) and (2),the integer solutions to (1) are given by

$$x = x(A,B) = 286A^2 + 836AB + 1430B^2$$
$$y = y(A,B) = 220A^2 - 176AB + 1100B^2$$
$$w = w(A,B) = 154A^2 + 506AB - 440B^2$$
$$z = z(A,B) = 88A^2 - 506AB - 770B^2$$
$$p = p(A,B) = 121A^2 - 605B^2$$

METHOD 4:

Introducing the linear transformations

$$v = X + 4T, p = X - 5T, u = 3U \tag{11}$$

in (3) , it leads to

$$U^2 = X^2 + 20T^2 \tag{12}$$

which is satisfied by

$$T = 2rs, X = 20r^2 - s^2, U = 20r^2 + s^2$$

Substituting the above values of X and T in (11), we have

$$u = 60r^2 + 3s^2, v = 20r^2 - s^2 + 8rs, p = 20r^2 - s^2 - 10rs$$

Substituting the above values of *u, v* and *p* in (2) , the corresponding non-zero integer values of x,y,z,w satisfying (1) are given by

$$x = 80r^2 + 8rs + 2s^2$$
$$y = 40r^2 - 8rs + 4s^2$$
$$z = -18rs$$
$$w = 40r^2 - 2rs - 2s^2$$

Note:

The linear transformation (11) can also be taken as

$$v = X - 4T, p = X + 5T, u = 3U$$

By following the procedure as above, a different set of integer solutions to (1) are obtained.

METHOD 5:

We write (12) as a system of double equations as shown in Table 2 below:

Table 2: System of Double Equations

System	I	II	III	IV	V
$U+X$	$10T^2$	$5T^2$	$2T^{2`}$	$T^{2`}$	$10T$
$U-X$	2	4	10`	20	2T

After some algebra, the solutions to (1) through solving each of the systems in Table 2 are exhibited as follows:

Solutions through system I:

$$x = 20k^2 + 4k + 2, y = 10k^2 - 4k + 4, z = -9k, w = 10k^2 - k - 2$$

Solutions through system II:

$x = 40k^2 + 8k + 4, y = 20k^2 - 8k + 8, z = -18k, w = 20k^2 - 2k - 4$

Solutions through system III:

$x = 4k^2 + 4k + 10, y = 2k^2 - 4k + 20, z = -9k, w = 2k^2 - k - 10$

Solutions through system IV:

$x = 8k^2 + 8k + 20, y = 4k^2 - 8k + 40, z = -18k, w = 4k^2 - 2k - 20$

Solutions through system V:

$x = 26k, y = 10k, z = -9k, w = 7k$

METHOD 6:

Write (3) as

$$5v^2 + 4p^2 = u^2 * 1 \tag{13}$$

Write 1 as

$$1 = \left(\frac{\left(\sqrt{5}b + i2\right)(\sqrt{5}\,b - i2)}{a^2} \right) \tag{14}$$

where

$$a^2 = 5b^2 + 4$$

Assume

$$u = 5\alpha^2 + 4\beta^2 \tag{15}$$

Using (14), (15) in (13) and applying the method of factorization, define

$$\sqrt{5}v + i2p = \left(\sqrt{5}\alpha + i2\beta\right)^2 \left(\frac{\sqrt{5}b + i2}{a} \right)$$

After performing a few calculations, we have

$$\left.\begin{aligned} u &= a^2(5P^2 + 4Q^2) \\ v &= a\left(5bP^2 - 4bQ^2 - 8PQ\right) \\ p &= a(5P^2 - 4Q^2 + 10bPQ \end{aligned}\right\} \tag{16}$$

Using (16) and (2), the values of x, y, z, w satisfying (1) are given by

$$x = 5a(a + b)P^2 + 4a(a - b)Q^2 - 8aPQ$$
$$y = 5a(a - b)P^2 + 4a(a + b)Q^2 + 8aPQ$$
$$z = 5a(1 - b)P^2 + 4a(b - 1)Q^2 + (10ab + 8a)PQ$$
$$w = 5a(1 + b)P^2 - 4a(1 + b)Q^2 + (10ab - 8a)PQ$$

For example ,when b=1 and a=3,the corresponding values of x,y,z,w satisfying (1) are as follows:

$$x = 60P^2 + 24Q^2 - 24PQ$$
$$y = 30P^2 + 48Q^2 + 24PQ$$
$$z = 54PQ$$
$$w = 30P^2 - 24Q^2 + 6PQ$$

Note: 1 on the R.H.S. of (13) may also be written as

$$1 = \left(\frac{\left(\sqrt{5}b + i\right)(\sqrt{5}\,b - i)}{a^2} \right)$$

where

$$a^2 = 5b^2 + 1$$

In this case, the corresponding values of x ,y ,z ,w satisfying (1) are found to be

$$x = 20a(a+b)P^2 + 4a(a-b)Q^2 - 8aPQ$$
$$y = 20a(a-b)P^2 + 4a(a+b)Q^2 + 8aPQ$$
$$z = 10a(1-2b)P^2 + 2a(2b-1)Q^2 + (20ab+8a)PQ$$
$$w = 10a(1+2b)P^2 - 2a(1+2b)Q^2 + (20ab-8a)PQ$$

For example when b=4 and a=9,the corresponding values of x,y,z,w satisfying (1) are as follows:

$$x = 9(260P^2 + 20Q^2 - 8PQ)$$
$$y = 9(100P^2 + 52Q^2 + 8PQ)$$
$$z = 9(-70P^2 + 14Q^2 + 88PQ)$$
$$w = 9(90P^2 - 18Q^2 + 72PQ)$$

III. CONCLUSION

In this paper, different sets of solutions in integers are exhibited to the cubic diophantine equation with four unknowns given by $x^3 - y^3 = 4(w^3 - z^3) + 3(x-y)^3$ through employing the linear transformations $x = u + v,\quad y = u - v,\quad w = p + v,\ z = p - v\quad u \neq v \neq p$. There may be other choices of transformations or methods to find some more infinite number of solutions. Since diophantine equations are rich in variety, one may try other choices of solutions for different cubic equations with multi-variables.

REFERENCES

[1]. L.E. Dickson, History of Theory of Numbers, Vol.2, Chelsea Publishing company, NewYork, 1952.

[2]. L.J. Mordell, Diophantine equations, Academic press, New York, 1969.

[3]. M.A. Gopalan, G. Sangeetha, “On the ternary cubic Diophantine equation $y^2 = Dx^2 + z^3$ ” , Archimedes J.Math 1(1), 2011, 7-14.

[4]. L.E. Dickson, History of Theory of Numbers, Vol.2, Chelsea Publishing company, NewYork, 1952

[5]. M.A. Gopalan, B. Sivakami, “Integral solutions of the ternary cubic equation $4x^2 - 4xy + 6y^2 = ((k+1)^2 + 5)w^3$ ”,Impact J.Sci.Tech, Vol.6, No.1, 2012, 15-22.

[6]. M.A. Gopalan, B. Sivakami, “On the ternary cubic Diophantine equation $2xz = y^2(x+z)$” , Bessel J.Math 2(3), 2012, 171-177.

[7]. S. Vidyalakshmi, T.R. Usharani, M.A. Gopalan, “Integral solutions of non-homogeneous ternary cubic equation $ax^2 + by^2 = (a+b)z^3$ ”, Diophantus J.Math 2(1), 2013, 31-38.

[8]. M.A. Gopalan, K. Geetha, “On the ternary cubic Diophantine equation $x^2 + y^2 - xy = z^3$ ”, Bessel J.Math., 3(2), 2013,119-123.

[9]. M.A. Gopalan, S. Vidhyalakshmi, A.Kavitha “Observations on the ternary cubic equation $x^2 + y^2 + xy = 12z^3$ ”, Antartica J.Math 10(5), 2013, 453-460.

[10]. M.A. Gopalan, S. Vidhyalakshmi, K. Lakshmi, “Lattice points on the non-homogeneous cubic equation $x^3 + y^3 + z^3 + (x+y+z) = 0$ ”, Impact J.Sci.Tech, Vol.7, No.1, 2013, 21-25.

[11]. M.A. Gopalan, S. Vidhyalakshmi, K. Lakshmi “Lattice points on the non-homogeneous cubic equation $x^3 + y^3 + z^3 - (x+y+z) = 0$ ”, Impact J.Sci.Tech, Vol.7, No1, 2013, 51-55,

[12]. M.A. Gopalan, S. Vidhyalakshmi, S. Mallika, “On the ternary non-homogenous cubic equation $x^3 + y^3 - 3(x+y) = 2(3k^2 - 2)z^3$ ”, Impact J.Sci.Tech, Vol.7, No.1, 2013, 41-45.

[13]. S. Vidhyalakshmi, M.A. Gopalan, S. Aarthy Thangam, “On the ternary cubic Diophantine equation $4(x^2 + x) + 5(y^2 + 2y) = -6 + 14z^3$ ” International Journal of Innovative Research and Review (JIRR), Vol 2(3)., pp 34-39, July-Sep 2014

[14]. M.A. Gopalan, N. Thiruniraiselvi and V. Kiruthika, “On the ternary cubic diophantine equation $7x^2 - 4y^2 = 3z^3$ ”, IJRSR, Vol.6, Issue-9, Sep-2015, 6197-6199.

[15]. M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, J. Maheswari, “On ternary cubic diophantine equation $3(x^2 + y^2) - 5xy + x + y + 1 = 12z^3$ ”, International Journal of Applied Research, 1(8), 2015, 209-212.

[16]. R. Anbuselvi, K. Kannaki, “On ternary cubic diophantine equation $3(x^2 + y^2) - 5xy + x + y + 1 = 15z^3$ ”, IJSR, Vol.5, Issue-9, Sep 2016, 369-375.

[17]. G. Janaki, C. Saranya, “Integral solutions of the ternary cubic equation $3(x^2 + y^2) - 4xy + 2(x + y + 1) = 972z^3$ ”, IRJET, Vol.04, Issue 3, March 2017, 665-669.

[18]. S. Vidhyalakshmi, T.R. Usha Rani, M.A. Gopalan, V. Kiruthika, “On the cubic equation with four unknowns $x^3 + y^3 = 14zw^2$ ”, IJSRP, Volume 5, Issue 3, March 2015, 1-11.

[19]. M.A. Gopalan, S. Vidhyalakshmi, G. Sumathi, "On the homogeneous cubic equation with four unknowns $X^3+Y^3=14Z^3-3W^2(X+Y)$", Discovery, 2(4), 2012, 17-19.

[20]. S. Vidhyalakshmi, M.A. Gopalan, A. Kavitha, "Observation on homogeneous cubic equation with four unknowns $X^3+Y^3=7^{2n}ZW^2$", IJMER, Vol.3, Issue 3, May-June 2013, 1487-1492.

[21]. M.A. Gopalan, S. Vidhyalakshmi, E. Premalatha, C. Nithya, "On the cubic equation with four unknowns $x^3+y^3=31(k^2+3s^2)zw^2$", IJSIMR, Vol.2, Issue 11, Nov-2014, 923-926.

[22]. M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, "On the cubic equation with four unknowns $x^3+4z^3=y^3+4w^3+6(x-y)^3$", International Journal of Mathematics Trends and Technology, Vol 20, No.1, April 2015, 75-84.

[23]. Dr. R. Anbuselvi, K.S. Araththi, "On the cubic equation with four unknowns $x^3+y^3=24zw^2$", IJERA, Vol.7, Issue 11 (Part-I), Nov-2017, 01-06.

[24]. E.Premalatha , M.A.Gopalan, "On Homogeneous Cubic Equation with Four Unknowns $x^3+y^3=13zw^2$", International Journal of Advances in Engineering and Management (IJAEM), Volume 2, Issue 2, 2020, Pp: 31-41.

S.Vidhyalakshmi, et. al. " A Search On the Integer Solutions of Cubic Diophantine Equation with Four Unknowns $\mathbf{x^3-y^3=4(w^3-z^3)+3(x-y)^3}$." *International Journal of Engineering and Science*, vol. 10, no. 08, 2020, pp. 13-18.