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Stability and Bifurcation Analysis for the Dynamical Model of a New Butterfly-shaped Chaotic Attractor

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Abstract:*Local dynamics including stability and Hopf bifurcation for the dynamical model of a new butterfly-shaped chaotic attractor is investigated both analytically and numerically in this paper. Equilibrium points and their stability conditions are presented. All the requirements for Hopf bifurcation are also stated. In addition, numerically simulations are devised to verify the analytical results.*

Keywords:*Stability, Hopf bifurcation, First Lyapunov coefficient.*

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I. INTRODUCTION

For the past three decades, scientists, engineers and physicists have paid an ernomous attention to chaotic nonlinear dynamical systems. Chaos has been an interesting phenomenon after Lorenz [1] discovered it in 1963. He proposed three autonomous differential equations to describe Rayleigh-Benard problem. Lorenz's chaotic system led many researchers into further investigations. Similar systems sprouted out of the investigations such as the Chen system [2]. In 2002, a new coined chaotic attractor was discovered by Lü and Chen, called the Lü system [3]. Liu et al [4]. also proposed a new three-dimensional autonomous chaotic system that consists of a nonlinear squared terms. Dias et al. [5] studied the existence of singular degenerate heteroclinic cycles for a suitable choice of the parameters at equilibrium E_+. Yang and Chen [6] proposed another three-dimensional chaotic system which has one saddle point and two stable equilibrium points. Zhang et al. [7] studied the Hopf bifurcation of a new chaotic system with chaos entanglement function. Chaotic system has been a paramount tool for nonlinear circuit analysis with different methods and Hopf bifurcation is one of the leading methods.

This paper seeks to investigate the local stability and Hopf bifurcation of a new butterfly-shaped chaotic attractor. The system proposed has three autonomous governing equations with three chaotic parameters, one multiplier xz and one quadratic term x^2 [8]. The dynamics of the system are studied both analytically and numerically. First Lyapunov coefficient method was futher used to determine the supercriticality and the subcritility of the system.

II. DYNAMIC MODEL OF THE NEW BUTTERFLY-SHAPED CHAOTIC ATTRACTOR SYSTEM

1.1 **Consider autonomous governing systems of the New Butterfly-shaped Chaotic Attractor below**

I.

$$\begin{cases} \dot{x} = a(x-y) \\ \dot{y} = xz + by \\ \dot{z} = -x^2 - cz \end{cases} \tag{1}$$

Where *x, y, z* are state variables of the system anda, b,c are parameters of the system.

2.1 Symmetric and invariance

From (1), the system is symmetric about the z-axis with the transformation, $(\bar{x}, \bar{y}, \bar{z}) \rightarrow (-\bar{x}, -\bar{y}, -\bar{z},)$. This implies that the system is invariant for all values of the parameters.

2.2 Equilibrium Points and Stability Analysis

Setting the right end of (1) to zero, the equation below is obtained

$$\begin{cases} a(x-y)=0 \\ xz+\mathrm{b}y=0 \\ -x^2-c\mathrm{z}=\ 0 \end{cases} \tag{2}$$

One can obtain equilibrium points for (2) as follows

$$\begin{cases} \mathrm{O}(0,0,0) \\ P_+(\sqrt{bc},\sqrt{bc},-b) \\ P_-(-\sqrt{bc},-\sqrt{bc},-b) \end{cases} \tag{3}$$

We first analyse the stability of the sysytem by linearizing it at $\mathrm{O}(0,0,0)$.
Let D_0 be the jacobian matrix of (2) evaluated at $\mathrm{O}(0,0,0)$

$$\mathrm{D}_0=\begin{bmatrix} -a & a & 0 \\ z & b & x \\ -2x & 0 & -c \end{bmatrix}=\begin{bmatrix} -a & a & 0 \\ 0 & b & x \\ 0 & 0 & -c \end{bmatrix} \tag{4}$$

The characteristic equation of the system corresponding to $\mathrm{O}(0,0,0)$ is

$$|\lambda I-D_0|=\ (\lambda+a)(\lambda-b)(\lambda+c)=\ 0 \tag{5}$$

$\lambda_1=-a, \lambda_2=b, \lambda_3=-c$

The equilibrium point is stable when $a, c\ >0$ and $b<0$ and unstable when $a<0$ and or $b>0$ or $c<0$
Hopf birfurcation does not occur at the equilibrium point $\mathrm{O}(0,0,0)$ due to the absence of pure imaginary pair of eigenvalues evolving from the characteristics equation.

Further stability is carried on the remaining two fixed points. The system is symmetric and invariant, we therefore considered the stability at P_+ only. Let D_+ be the jacobian matrix evaluated at the equilibrium point $\mathrm{P}_+(\sqrt{\mathrm{bc}},\sqrt{\mathrm{bc}},-\mathrm{b})$.

$$\mathrm{D}_+=\begin{bmatrix} -a & a & 0 \\ z & b & x \\ -2x & 0 & c \end{bmatrix}=\begin{bmatrix} -a & a & 0 \\ -b & b & \sqrt{bc} \\ -2\sqrt{bc} & 0 & c \end{bmatrix} \tag{6}$$

The characteristic equation corresponding to D_+ is

$$\left(\lambda^3+(a+c-b)\lambda^2+(ac-bc-ab-az)\lambda\ +(-2abc-acz-2ax^2)\right)=0 \tag{7}$$

The characteristic equation corresponding to the quilibrium point $\mathrm{P}_+(\sqrt{\mathrm{bc}},\sqrt{\mathrm{bc}},-\mathrm{b})$ is

$$\lambda^3+(a+c-b)\lambda^2+(ac-bc)\lambda-2abc\ =\ 0 \tag{8}$$

By Routh-Hurwitz criterion , P_+ is stable if and only if the following conditions are satisfied

$$\begin{cases} a+c-b>0 \\ (a+c-b)(ac-bc)>0 \end{cases} \tag{9}$$

2.3 **Hopf Bifurcation**

According to hopf bifurcation theory [9], the characteristic equations corresponding to P_+ and P_- has purely imaginary eigenvalues. This gives an indication that if Eg.(8) further satisfies the following conditions, hopf bifurcation may likely occur.

$$\begin{cases} Re(\lambda)|_{c=c_0}=0 \\ Re(\lambda)|_{c=c_0}\neq 0 \\ \dfrac{d}{db}Re(\lambda)|_{c=c_0}\neq\ 0 \end{cases} \tag{10}$$

Where, c_0 is the critical value of of c for Hopf bifurcation

Letting $\lambda_2=i\omega, \lambda_3=-i\omega$, where $\omega>0$, substituting $\lambda_2=i\omega$ into Eq.(9), we obtain

$$(i\omega)^3+(a+c-b)(i\omega)^2+(ac-bc)(i\omega)-2abc\ =\ 0 \tag{11}$$

Separating the real and the imaginary parts of Eq.(11) yields

$$\begin{cases} \omega^2=(bc-ac) \\ \omega^2=-\dfrac{2abc}{a+c-b} \end{cases} \tag{12}$$

With further conditions as follows

$$\begin{cases} (bc-ac)>0 \\ -\dfrac{2abc}{a+c-b}>0 \\ a+c-b>0 \end{cases} \tag{13}$$

Also, c_0 can be derived as

$$c_0 = \frac{-a^2 + 4ba - b^2}{-b + a} \tag{14}$$

The eigenvalues corresponding to it are

$$\lambda_1 = -(a + c - b), \lambda_{2,3} = \pm i\sqrt{ac - bc} \tag{15}$$

From Eq.(10), when c is bigger than c_0, the equilibrium point P_+ is stable. Once c is smaller than c_0, it becomes stable.

The first Lyapunov coefficient [10] is used to discuss the supercriticality or subcritility of the Hopfbifurcation. Let C^n be n-dimensional complex Hilbert space with inner product.

$$<x, y> = \sum_{i=1}^{n} x_i y_i\,, \quad \text{for } x = (y_1, y_2, \ldots y_n)^T, x_i y_i \in C(i = 1,2, \ldots n),$$

Introduce norm $\|x\| = \sqrt{<x,x>}$ and C^n is a Hilbert space.

$$\dot{x} = Ax + F(x), x \in R^n \tag{16}$$

Where

$$F(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O \tag{17}$$

$B(x,x)$ and $C(x,x,x)$ are bilinear and trilinear functions respectively.

Also, we introduce the adjoint eigenvector p$\in C^n$ which satisfies the following following condions.

$$M^T p = -i\omega p, M^T \bar{p} = i\omega\bar{p} \quad and \quad <p,q> = 1 \tag{18}$$

When $a = 2, b = 1, then\ c_0 = 3, x = 0, y = 0, z = 0$

The Jacobian matrix of Eq.(2) evaluated at P_+ is

$$D_+ = \begin{bmatrix} -2 & 2 & 0 \\ -1 & 1 & \frac{\sqrt{15}}{5} \\ -2\frac{\sqrt{15}}{5} & 0 & \frac{3}{5} \end{bmatrix} \tag{19}$$

The first Lyapunov coefficient method of the system (2) at the equilibrium point P_+ is written as[9]

$$\begin{aligned} l_1(0) = \frac{1}{2}Re(<p, C(q,q,\bar{q})>) - 2<p, B\big(q, M^{-1}B(q,\bar{q})\big)> \\ + <p, B\big(\bar{q}(2i\omega E - M)^{-1}B(q,q)\big)> \end{aligned} \tag{20}$$

Next, we calculate the corresponding vector $<p,q>$ of the matrix D_+ as in Eq,(7). After complex computations, we have the vector quantity $<p,q>$ corresponding to the matrix D_+ that satisfy

$Mq = i\omega q, M^T p = -i\omega p \quad and \quad <p,q> = 1.$

For

$$q = \begin{vmatrix} -\frac{5\left(127 + 18\sqrt{43}\right)}{2} - \frac{1125i\left(\sqrt{43}\right)}{6} \\ -\frac{5\left(127 + 18\sqrt{43}\right)}{2} + \frac{15i\left(\sqrt{43}\right)}{6} \\ -\frac{5\left(127 + 18\sqrt{43}\right)}{2} - \frac{1125i\left(\sqrt{43}\right)}{6} \end{vmatrix} \tag{21}$$

$$p = \begin{vmatrix} -\frac{450\left(127 + 18\sqrt{43}\right)}{2} - \frac{1125i\left(\sqrt{43}\right)}{6} \\ -\frac{150\left(127 + 18\sqrt{43}\right)}{2} + \frac{15i\left(\sqrt{43}\right)}{6} \\ -\frac{375\left(127 + 18\sqrt{43}\right)}{2} - \frac{1125i\left(\sqrt{43}\right)}{6} \end{vmatrix} \tag{22}$$

$$\bar{q} = \begin{bmatrix} -\frac{450(127+18\sqrt{43})}{2} + \frac{1125i(\sqrt{43})}{6} \\ -\frac{150(127+18\sqrt{43})}{2} - \frac{15i(\sqrt{43})}{6} \\ -\frac{375(127+18\sqrt{43})}{2} + \frac{1125i(\sqrt{43})}{6} \end{bmatrix} \quad (23)$$

Which satisfy $Mq = i\omega q, M^T p = -i\omega p \;\; and \;\; <p, q> = 1$.
For system (2), the bilinear and trilinear functions are

$$B(X, X') = (yz', xz', xy'), C(X, X', X'') = (0,0,0)^T \quad (24)$$

$$<p, B\left(q, M^{-1}B(q, \bar{q})\right)> = -12.33529321 + 21.033974i \quad (25)$$

$$<p, B\left(\bar{q}(2i\omega E - M)^{-1}B(q, q)\right)> = -13.32215292 - 3.734022300i \quad (26)$$

Consequently, we have

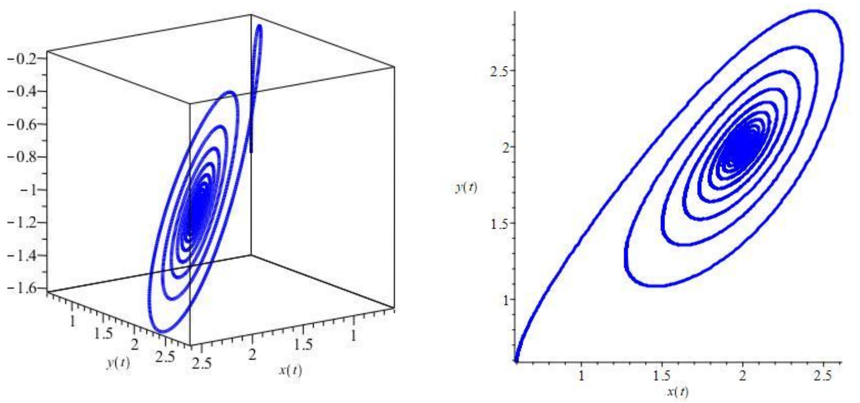
$$l_1(0) = \frac{1}{2}Re(<p, C(q, q, \bar{q})>) - 2<p, B\left(q, M^{-1}B(q, \bar{q})\right)> + <p, B\left(\bar{q}(2i\omega E - M)^{-1}B(q, q)\right)> = -25.9574 \quad (27)$$

Base on the sign of the first Lyapunov coeeficient, the Hopf bifurcation is supercritical.

III. NUMERICAL SIMULATIONS

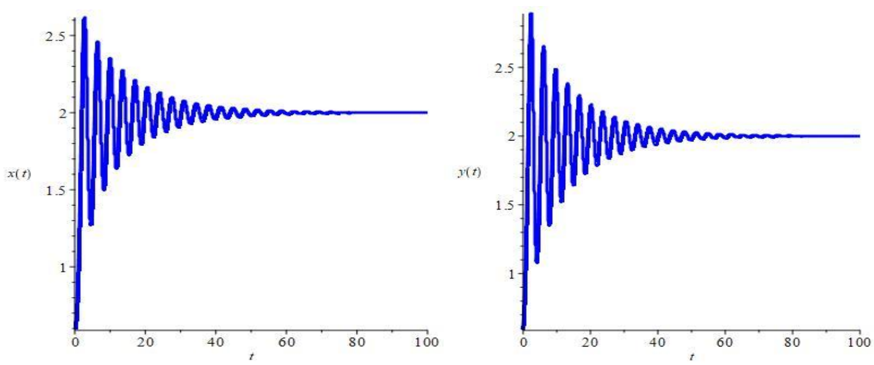
Numerical results are obtained by using fourth-order Runge-Kutta method in this section. We fix $a = 2$and $b = 1$,to obtain the phase potraits and trajectories as in Figures 1-4. For$c > c_0$, Figure 1(a) and (b), shows the phase potrait (x(t) y(t) z(t)) and (x(t) y(t)) respectively whiles Figure 2(a) and 2(b) shows the trajectories of (x(t)) and (y(t)) respectively. It could be seen that stable for $c > c_0$ the equilibrium point is stable.

Similarly, for$c < c_0$,Figure 3(a) and (b), shows the phase potrait (x(t) y(t) z(t)) and (x(t) y(t)) respectively whiles Figure 4(a) and (b) shows the trajectories of (x(t)) and y(t) respectively. When $c < c_0$, The system generate hopf bifurcation and it could be seen from the above that the results agree with the analytical one.



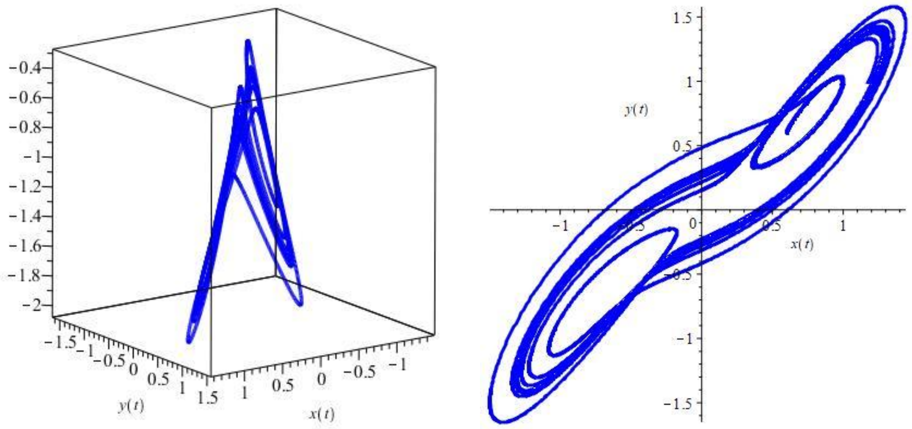
(a) Phase potrait of *x(t) y(t) z(t)*(b)Phase potrait of x(t) y(t)

Figure 1 Phase Potrait for *a*=2, *b*=1 and *c*=4



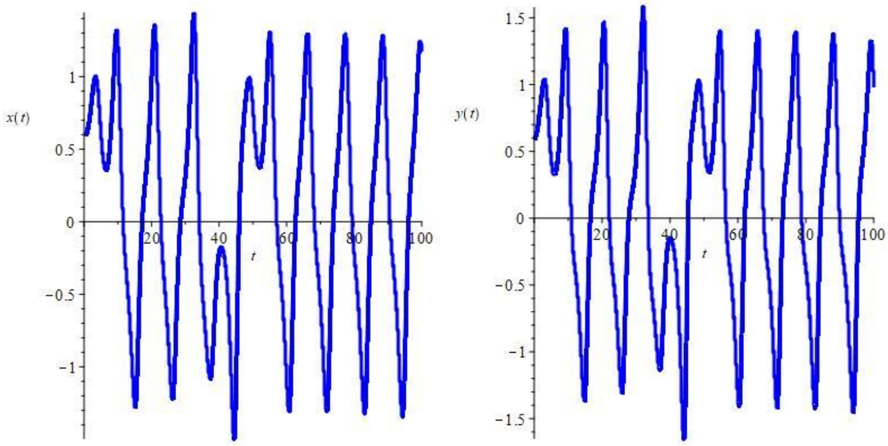
(a) Trajectories of *x(t)* (b)Trajectories of *y(t)*

Figure 2 Trajectoriefor *a*=2, *b*=1 and *c*=4



(a) Phase Potrait of *x(t) y(t) z(t)* (b) Phase Potrait of *x(t) y(t)*

Figure 3 Phase Potrait for a=2, b=1 and c= 0.6



(a)Trajectories of *x(t)* (b)Trajectories of y(t)

Figure 4 Trajectoriefor *a*=2, *b*=1 and *c*=0.6

IV. CONCLUSION

Local dynamics including stability and Hopf bifurcation for the dynamical model of the new butterfly-shaped chaotic attractor is investigated both analytically and numerically in this paper. Hopf bifurcation theorem and the first Lyapunov coefficient method are used to investigate the conditions and types of bifurcation in the system. From the numerical analysis,it is presented that the Hopf bifurcation of the system is supercritical.

The results provide some guidance for nonlinear circuit designs.

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